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BALLISTICS OF LONG-RANGE GUIDED ROCKETS

R. F. Appazov, et al

Foreign Technology Division
Wright-Patterson Air Force Base, Ohio

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AIR FORCE SYSTEMS COMMAND

BALLISTICS OF LONG-RANGE GUIDED ROCKETS

by

R. F. Appazov, S. S. Lavrov and
V. P. Mishin

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By: R. F. Appazov, S. S. Lavrov and V. P. Mishin

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R. F. Appazov, S. S. Lavrov.
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ABSTRACT: This book serves as an introduction to the study of the ballistics of long-range missiles. It discusses flight theory and methods of calculating trajectories. The author expresses appreciation to P. P. Karaulov, S. S. Rozanov, and M. S. Florianskiy for their assistance in preparing various paragraphs of the book. There are 13 references, all Soviet.

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Dedicated to the Memory
of Academician Sergey
Pavlovich Korolev

PREFACE

Expounded in the book are certain theoretical bases and the most important practical methods of investigation and calculation of motion of the center of masses of guided long-range ballistic rockets.

The book is intended basically for those who for the first time are encountered with the ballistics of long-range rockets. Therefore, the authors have tried as far as possible to give a presentation on all problems of ballistic character with which one must encounter in the process of the designing of rockets. Along with an account of questions of the theory of flight and methods of calculation of the trajectory, the reader will find in the book the formulation of a number of problems whose development can be of considerable interest. These are basically problems referring to the selection of the form of trajectory and to methods of control of the flight range of rockets.

The book consists of four parts.

In the first part there is conducted an analysis of forces and moments having an effect on the rocket; there are general equations of motion; there is investigated the possibility of their simplification depending upon the character of the problem to be solved and, finally, integration is conducted of equations of motion for free flight in a vacuum and the solution of these equations is investigated.

In the second part questions are examined connected with the practical solution of the basic problem of ballistics and problem of designing: the method of calculation of the trajectory and composition of preliminary tables of firing and an analysis of the influence of basic design parameters on flying characteristics.

The third part is devoted to dispersion with the firing by long-range rockets and adjacent questions, in particular, the influence of certain peculiarities of the control system and propulsion system on the accuracy of fire.

Examined in the fourth part is the problem on the selection of the so-called pitch-law program of the change in angle of inclination of the axis of the rocket. The pitch program determines the form of the powered-flight trajectory and thereby influences both the flying range of the rocket, its other flying characteristics, including the accuracy of firing.

This book can serve as an aid for students of higher educational institutions and engineers specializing in the field of ballistics of rockets.

Contributing to seperate paragraphs of the book were P. P. Karaulov and S. S. Rozanov; a number of useful remarks were made by M. S. Florianskiy. The authors are grateful to all of them for the help rendered.

INTRODUCTION

By long-range rocket (РДД) we mean a controlled aircraft with a reactive engine intended for the transfer of a payload at great distances on a preassigned trajectory, a greater part of which passes in very rarefied layers of the atmosphere.

Long-range rockets possess a number of peculiarities separating them in an independent class of aircraft. The dynamics of their flight has much in common with the dynamics of flight of aircraft, artillery shells, unguided rockets, but at the same time it obeys in many details its special regularities and therefore requires independent investigation.

The РДД trajectory consists of two sharply differentiated sections. On the first section, which is called the powered section, the rocket collects kinetic energy. By the quantity of accumulated kinetic energy at the end of the powered section the РДД sharply differs from other transport means. Having a mass of payload of the same order as that of a bomber aircraft, the РДД attains a speed considerably exceeding the speed of artillery shells. But this speed is gathered by the rocket gradually and is attained in greatly rarefied layers of the atmosphere, which permits bringing to a minimum of the expenditure of energy on surmounting atmospheric drag. The quantity of accumulated kinetic energy is the most important index of the РДД perfection.

On the second section, called the free-flight trajectory or the free ballistic path, the accumulated energy is used for transportation of payload at a great distance. According to the character of the use of the РДД energy, it is possible to divide rockets into two basic groups:

a) ballistic rockets flying after the turning off of the engine similar to artillery shells and controlled only prior to the moment of the turning off of the engine;

b) glide rockets, controlled during the period of the entire trajectory, which use aerodynamic lift to increase the flying range.

In this book only ballistic РДД are examined.

Long-range rockets, just as artillery shells, fly on trajectories assigned them before launch. But, in contrast to artillery shells, the РДД are controlled in flight, enabling the possibility to a considerable degree to compensate the influence of a number of causes having an effect on the powered section and leading to a deviation of the actual trajectory from the assigned. The control system of the ballistic РДД solves the following problems:

a) maintaining the assigned, gradually variable during flight, orientation of axes of the rocket in space (control of motion around the center of gravity);

b) maintaining the assigned direction of flight and form of trajectory and also of the given value and direction of the speed of flight (control of the motion of the center of gravity);

c) turning off of the engine at the moment when the kinematic parameters of motion of the center of gravity of the rocket (speed, its direction and coordinate of the center of gravity) in totality provide flight for the assigned distance (range control of the flight).

The control system thus provides flight of the rocket in accordance with the performed aiming and setting of the control instruments, but the very problems of aiming or guiding of the rocket at the target are not solved by it.

After turning off of the engine the greater part of the flight of the ballistic rocket occurs in practically a vacuum under action of forces not controllable but those which are exactly well-known. This, on the one hand, excludes the possibility of control on the greater part of the section of free flight and on the other hand, increases the accuracy of firing.

The named peculiarities of the РДД determine the specific character of ballistics — the theory of their motion. On the powered flight trajectory the motion of the rocket should be examined taking into account, first, the great speed of change of the mass of the rocket and, secondly, the presence of control. The first circumstance makes the laws of mechanics of bodies and systems of constant mass inapplicable in final form to the study of motion of the rocket, and the second compels one to examine the motion of the center of gravity of the rocket jointly with the motion of the rocket around the center of gravity. It should be noted that just as in ballistics of artillery shells the motion of a rocket around the center of gravity is examined neglecting the small oscillations around the center of gravity. Bases for this are even greater that the control system should extinguish oscillations of the rocket.

In examining the motion on the section of free flight, due to the great distance, altitude and speed of the flight, it is necessary to consider a change in acceleration of terrestrial gravity in magnitude and direction and the influence of rotation of the earth. But then the investigation of the trajectory is facilitated by the small magnitude of aerodynamic forces on the free ballistic path and at the end of the powered section. There appears the possibility of methods of approximation of the calculation which possess high enough accuracy but at the same time are simple.

Ballistics of РДД should solve the following problems:

1. Determination of the trajectory and other basic characteristics of the motion of the rocket with well-known design parameters and control system with assigned sighting data (direct basic problem) or, under those same conditions, determination of sighting data from assigned launching points and the target (inverse problem).

2. Selection of the form of trajectory providing the best use of possibilities of the rocket (selection of the control program).

3. Investigation of the dependence of flying characteristics of the rocket, in the first place, range of its flight, on the design parameters for the purpose of selection of most advantageous combination of these parameters (ballistic designing).

4. Investigation of the influence of different perturbing factors — scattering of design parameters, change in external conditions of flight, errors in control instruments — on flying characteristics of the rocket (investigation of dispersion and related questions).

These problems are closely connected with the solution of a number of other problems related to aerodynamics (determination of the magnitude of aerodynamic forces and thermal regime of construction depending upon the selected trajectory), dynamics of construction (design of elastic oscillations and oscillations of liquid in the tanks), theory of automatic control (investigation of processes of stabilization and stability of motion of the rocket on its calculated trajectory, selection of laws of control), calculation of design of the rocket for strength (determination of loads on the construction and their dependence on the flight path), and other disciplines. The great role of ballistics in the solution of design problems is very great: the selection of the configuration of the rocket, its design and values of its constructive and power characteristics, which in the very best manner correspond to requirements presented to the given rocket. All these adjacent questions are partially and briefly touched upon in the book only in connection with the solution of the above-mentioned problems of ballistics. The very problems of ballictics, which are thus reduced to the investigation of the undisturbed motion of the center of masses of rocket, are examined neglecting many, sometimes very important, details in order to pay attention to the basic peculiarities of these problems, the regularities with which they obey, and methods of their solution. Knowledge of these methods and regularities will allow an engineer to begin independent work in the field of ballistics of rockets.

PART ONE

GENERAL THEORY OF MOTION

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CHAPTER I

SYSTEMS OF COORDINATES

§ 1. Terrestrial Coordinates

We will examine the motion of rocket in the rectangular coordinate system Oxyz, which is motionlessly connected with earth (Fig. 1.1). This system of coordinates will be called terrestrial. The axis Ox of the terrestrial system will be directed along the tangent to the surface of earth at the point of launch in the direction of aiming, axis Oy — vertically upwards at the point of start, and axis Oz — in such a way to obtain the right-handed system, i.e., perpendicular to the plane Oxy to the right of the direction of aiming. We consider earth a sphere with a radius R = 6,371,110 m (the volume of such a sphere is equal to the volume of a terrestrial spheroid). The point with coordinates (x, y, z) has with respect to the center of the earth radius vector

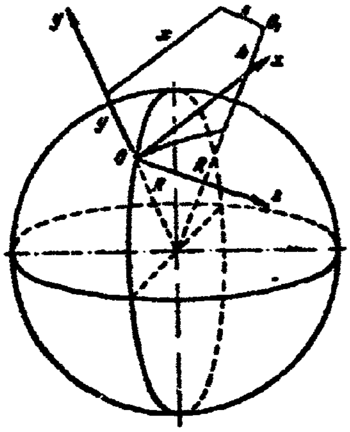

Fig. 1.1.

$$\mathbf{r} = x\mathbf{x}^0 + (R+y)\mathbf{y}^0 + z\mathbf{z}^0; \qquad (1.1)$$

the length of the vector

$$r = \sqrt{(R+y)^2 + x^2 + z^2} \qquad (1.2)$$

is the distance of this point from the center of the earth. The altitude of this point above the surface of the earth is equal to

$$h = r - R = \sqrt{(R+y)^2 + x^2 + z^2} - R. \qquad (1.3)$$

The terrestrial system of coordinates is not inertial, since it participates in the rotation of the earth around its axis, accomplishing a full revolution in one stellar day (86,164 s). The angular velocity of rotation is equal to

$$\omega_3 = \frac{2\pi}{86\,164} = 7.2921 \cdot 10^{-5} \frac{1}{s}. \qquad (1.4)$$

The vector of the angular velocity of the earth $\boldsymbol{\omega}_3$ is directed along the axis of rotation from the south pole to the north, since the rotation of earth occurs from west to east. If we designate the geographic latitude of the point of launch by φ_r, then vector $\boldsymbol{\omega}_3$ can be decomposed into two components (Fig. 1.2): vertical, directed along the axis Oy.

$$\omega_{3_y} = \omega_3 \sin \varphi_r.$$

and horizontal, directed in the plane Oxz on the tangent to the meridian:

$$\omega_{3_N} = \omega_3 \cos \varphi_r.$$

The horizontal component in turn can be decomposed on axes Ox and Oz into components

$$\omega_{3_x} = \omega_3 \cos \varphi_r \cos \psi,$$
$$\omega_{3_z} = -\omega_3 \cos \varphi_r \sin \psi.$$

Thus the vector of angular velocity of the earth can be represented in the form

$$\omega_3 = \omega_3 (\cos \varphi_r \cos \psi x^0 + \sin \varphi_r y^0 - \cos \varphi_r \sin \psi z^0). \qquad (1.5)$$

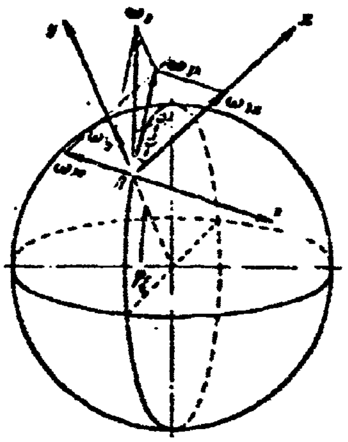

Fig. 1.2.

Motion of the center of gravity of earth in space will be considered rectilinear and uniform, disregarding the curvature of the terrestrial orbit.

The point moving relative to the terrestrial system of coordinates, with moving coordinates in this system x, y, z, has a relative speed of

$$v = \dot{x} x^0 + \dot{y} y^0 + \dot{z} z^0 \qquad (1.6)$$

and relative acceleration

$$J_r = \ddot{x} x^0 + \ddot{y} y^0 + \ddot{z} z^0. \qquad (1.7)$$

Absolute acceleration of this point will be equal to

$$J = J_r + J_e + J_c, \qquad (1.8)$$

where J_e is the translational acceleration equal to $J_e = \omega_3 \times (\omega_3 \times r) = \omega_3 (\omega_3 \cdot r) - r\omega_3^2$, $J_c = 2\omega_3 \times v$ is the Coriolis acceleration.

Using expressions (1.1), (1.5) and (1.6), we will find the decomposition of vectors J_e and J_c about axes of the terrestrial system of coordinates. The scalar product $\omega_3 \cdot r$ can be given in the form

$$\omega_3 \cdot r = \omega_3 r_\omega,$$

where

$$r_\omega = x \cos \varphi_r \cos \psi + (R + y) \sin \varphi_r - z \cos \varphi_r \sin \psi \qquad (1.9)$$

is the projection of radius vector r on the axis of rotation of the earth.

Consequently,

$$J_e = \omega_3 r_\omega \omega_3 - \omega_3^2 r = \omega_3^2 [(r_\omega \cos \varphi_r \cos \psi - x) x^0 + \\ + (r_\omega \sin \varphi_r - R - y) y^0 + (-r_\omega \cos \varphi_r \sin \psi - z) z^0] \qquad (1.10)$$

and

$$J_c = 2\begin{vmatrix} x^0 & y^0 & z^0 \\ \omega_3\cos\varphi_r\cos\psi & \omega_3\sin\varphi_r & -\omega_3\cos\varphi_r\sin\psi \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} =$$

$$= 2\omega_3[(\dot{y}\cos\varphi_r\sin\psi + \dot{z}\sin\varphi_r)x^0 +$$

$$+(-\dot{x}\cos\varphi_r\sin\psi - \dot{z}\cos\varphi_r\cos\psi)y^0 +$$

$$+(-\dot{x}\sin\varphi_r + \dot{y}\cos\varphi_r\cos\psi)z^0]. \qquad (1.11)$$

§ 2. Bound Coordinates

Besides the terrestrial system of coordinates, we will use the rectangular system of coordinates $O_1x_1y_1z_1$ (Fig. 2.1) connected with the rocket. In short we will call this system bound. We place the origin of the bound system of coordinates at the center of gravity of the rocket and direct the axis O_1x_1 along the longitudinal axis of the rocket toward its summit. At launch the rocket is set vertically, and therefore at the time of launching axis O_1x_1 coincides in direction with axis Oy of the terrestrial system of coordinates. Axis O_1z_1 will be directed in such a manner that it at that same moment is parallel to axis Oz; then axis O_1y_1 will take a direction opposite the direction of the axis Ox. In other words, the direction of axes of the bound system of coordinates at the time of launch will coincide with directions of corresponding axes of the terrestrial system if one were to turn the latter at an angle of 90° around axis Oz in a direction from axis Ox to axis Oy.

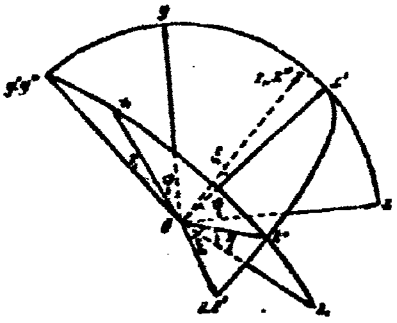

Fig. 2.1.

In flight the directions of axes of the bound system of coordinates with respect to the terrestrial are changed. We will determine then by angles of three turns, which it is necessary to produce in definite order in order to combine directions of axes of the terrestrial system with directions of axes of the bound system of coordinates.

Since we are now interested only in the direction of axes which are not changed with parallel translation, then we preliminarily combine by means of parallel translation the origin of the terrestrial system of coordinates O with the origin of the bound system O_1. After that we will perform the following operations:

1. Let us turn the terrestrial system at angle φ around axis Oz in such a manner that the plane Oxz passes through axis Ox_1; the obtained system will be designed Ox'y'z'.

2. With a turn around axis Oy' at angle ξ we combine axis Ox' with axis Ox_1; the obtained system will be designated Ox"y"z".

3. With a turn around axis Ox" at angle η we combine axes Oy" and Oz" with axes Oy_1 and Oz_1.

As a result of these three turns, the terrestrial system will be combined with the bound system. Let us find the formulas of transition from one system of coordinates to the other. The transition from system Oxyz to system Ox'y'z' is expressed by formulas

$$\left.\begin{aligned} x' &= x\cos\varphi + y\sin\varphi,\\ y' &= -x\sin\varphi + y\cos\varphi,\\ z' &= z.\end{aligned}\right\} \qquad (2.1)$$

Formulas of transition from system Ox'y'z' to system Ox"y"z" have the form

$$\left.\begin{aligned} x'' &= x'\cos\xi - z'\sin\xi,\\ y'' &= y',\\ z'' &= x'\sin\xi + z'\cos\xi.\end{aligned}\right\} \qquad (2.2)$$

Finally, the transition from system Ox"y"z" to system $Ox_1y_1z_1$ is carried out by means of formulas

$$\left.\begin{aligned} x_1 &= x'',\\ y_1 &= y''\cos\eta + z''\sin\eta,\\ z_1 &= -y''\sin\eta + z''\cos\eta.\end{aligned}\right\} \qquad (2.3)$$

Substituting expressions (2.1) for x', y', z' into formulas (2.2), we obtain:

$$\begin{aligned} x'' &= x\cos\varphi\cos\xi + y\sin\varphi\cos\xi - z\sin\xi,\\ y'' &= -x\sin\varphi + y\cos\varphi,\\ z'' &= x\cos\varphi\sin\xi + y\sin\varphi\sin\xi + z\cos\xi.\end{aligned}$$

If these expressions for x", y", z" are substituted in formulas (2.3), then we will find the formulas interesting to us of the direct transition from the terrestrial system Oxyz to the bound system $O_1x_1y_1z_1$ (on the above-mentioned condition that the origins of both systems of coordinates O and O_1 are combined):

$$\left.\begin{aligned} x_1 &= x\cos\varphi\cos\xi + y\sin\varphi\cos\xi - z\sin\xi,\\ y_1 &= x(-\sin\varphi\cos\eta + \cos\varphi\sin\xi\sin\eta) +\\ &\quad + y(\cos\varphi\cos\eta + \sin\varphi\sin\xi\sin\eta) + z\cos\xi\sin\eta,\\ z_1 &= x(\sin\varphi\sin\eta + \cos\varphi\sin\xi\cos\eta) +\\ &\quad + y(-\cos\varphi\sin\eta + \sin\varphi\sin\xi\cos\eta) + z\cos\xi\cos\eta.\end{aligned}\right\} \qquad (2.4)$$

The geometric meaning of angles φ, ξ and η is the following: the angle φ determines the position of the inclined plane perpendicular to plane Oxy and passing through the longitudinal axis of the rocket, the angle ξ is the angle (in this inclined plane) between the longitudinal axis of the rocket and plane Oxy, and finally, the angle η is the angle of rotation of the rocket about the longitudinal axis. It is accepted to call φ the pitch angle, ξ – the yaw angle, and η – the roll angle.

One of the problems of the control system of the flight of the rocket is that in order not to allow the appearance of great values of angles ξ and η, and to change angle φ according to the assigned law defined beforehand.

Since we examine the normal flight of the rocket with a properly working control system, we will consider angles ξ and η small and replace their cosines with unity, and the sines with the angles themselves:

$$\cos\xi = \cos\eta = 1, \quad \sin\xi = \xi, \quad \sin\eta = \eta.$$

Producing such replacement in formulas of transition (2.4) and rejecting terms containing the product of small quantities ξ and η, we will obtain the following simplified formulas which we will use henceforth:

$$\left.\begin{aligned} x_1 &= x\cos\varphi + y\sin\varphi - z\xi, \\ y_1 &= -x\sin\varphi + y\cos\varphi + z\eta, \\ z_1 &= x(\xi\cos\varphi + \eta\sin\varphi) + y(\xi\sin\varphi - \eta\cos\varphi) + z. \end{aligned}\right\} \qquad (2.5)$$

Coefficients in these formulas are cosines of angles between axes of the terrestrial and bound systems of coordinates or the so-called direction cosines (Table 2.1).

Table 2.1

	Ox	Oy	Oz
O_1x_1	$\cos\varphi$	$\sin\varphi$	$-\xi$
O_1y_1	$-\sin\varphi$	$\cos\varphi$	η
O_1z_1	$\xi\cos\varphi + \eta\sin\varphi$	$\xi\sin\varphi - \eta\cos\varphi$	1

If components of a certain vector **A** on axes of the terrestrial system of coordinates are equal to A_x, A_y, A_z, then on axes of the bound system this vector has the following components:

$$\left.\begin{aligned} A_{x_1} &= A_x\cos\varphi + A_y\sin\varphi - A_z\xi, \\ A_{y_1} &= -A_x\sin\varphi + A_y\cos\varphi + A_z\eta, \\ A_{z_1} &= A_x(\xi\cos\varphi + \eta\sin\varphi) + A_y(\xi\sin\varphi - \eta\cos\varphi) + A_z. \end{aligned}\right\} \qquad (2.6)$$

Conversely, the vectorial components in the terrestrial system are expressed in terms of components in the bound system by means of such formulas:

$$\left.\begin{aligned} A_x &= A_{x_1}\cos\varphi - A_{y_1}\sin\varphi + A_{z_1}(\xi\cos\varphi + \eta\sin\varphi), \\ A_y &= A_{x_1}\sin\varphi + A_{y_1}\cos\varphi + A_{z_1}(\xi\sin\varphi - \eta\cos\varphi), \\ A_z &= -A_{x_1}\xi + A_{y_1}\eta + A_{z_1}. \end{aligned}\right\} \qquad (2.7)$$

During the flight of the rocket angles φ, ξ and η do not remain constant. Let us designate their derivatives, as usual, by $\dot{\varphi}$, $\dot{\xi}$ and $\dot{\eta}$, and let us find the form of nonholonomic constraint among these derivatives and projections ω_{x_1}, ω_{y_1}, ω_{z_1} of the angular velocity of the rocket on the axis of the bound system of coordinates. Vector $\dot{\varphi}$ is directed along the axis Oz of the terrestrial system of coordinates. Its direction cosines coincide with coefficients at z in equations (2.4) and in simplified form are contained in the last column of Table 2.1. Vector $\dot{\xi}$ is directed along the intermediate axis Oy' (Oy") lying in plane Oy_1z_1 and generator angle η with the axis Oy_1 and angle 90° + η with the axis Oz_1. Consequently, its direction cosines in the bound system of coordinates $Ox_1y_1z_1$ will be (0, cos η, -sin η). Finally, the vector $\dot{\eta}$ is directed along the axis Ox_1 and has direction cosines (1, 0, 0). Consequently, the nonholonomic constraint interesting to us has the form

$$\left.\begin{aligned} \omega_{x_1} &= -\dot{\varphi}\sin\xi + \dot{\eta}, \\ \omega_{y_1} &= \dot{\varphi}\cos\xi\sin\eta + \dot{\xi}\cos\eta, \\ \omega_{z_1} &= \dot{\varphi}\cos\xi\cos\eta - \dot{\xi}\sin\eta. \end{aligned}\right\} \qquad (2.8)$$

or, in simplified form,

$$\left.\begin{array}{l} \omega_{\xi}=-\dot{\psi}\vartheta+\dot{\varphi} \\ \omega_{\eta}=\dot{\psi}\varphi+\dot{\vartheta} \\ \omega_{\zeta}=\dot{\psi}-\dot{\vartheta}\varphi \end{array}\right\} \qquad (2.9)$$

CHAPTER II

FORCES AND MOMENTS ACTING ON THE ROCKET

By rocket as a mechanical system we will imply all those masses which at the given moment of time are included in the volume limited by the external surface of body and control surfaces of the rocket and by the plane of exit section of the nozzle (or nozzles) of the engine.

On the rocket the following external forces act: gravity, aerostatic and aerodynamic forces and forces from controls. Gravity is the mass force, i.e., is composed of elementary forces applied to each particle of mass of the rocket. The remaining forces which are surface, namely, aerostatic and aerodynamic forces, are composed of elementary forces applied to each elementary area of the body surface of the rocket, and forces from controls in this way are composed of elementary forces on the surface of the control surfaces.

Let us proceed to the investigation of these forces and moments.

§ 3. Gravity

Gravity, or the weight of rocket, G, is expressed by well-known formula

$$G = mg. \tag{3.1}$$

The mass of the rocket m is determined by operating conditions of the engine (flow rate per second) from switching on of the engine prior to the examined moment of time. If by $\dot{m}$ we designate the flow rate per second of mass through the nozzle exit section, i.e., the absolute value of the derivative mass in time:

$$\dot{m} = \left|\frac{dm}{dt}\right| = -\frac{dm}{dt}, \tag{3.2}$$

then for the mass of the rocket at the time t will be obtained by the following expression:

$$m = m_{\text{вкл}} - \int_{t_{\text{вкл}}}^{t} \dot{m}\,dt, \tag{3.3}$$

where $t_{\text{ВКЛ}}$ is the moment of the switching on of the engine prior to which the mass of rocket is not changed and is equal to $m_{\text{ВКЛ}}$.

The flow rate per second, in general, is inconstant. Considerable changes in the flow rate occur in transient conditions of the operation of the engine (switching on, switching to a smaller thrust, complete turning off). But also during operation of the engine in the steady-state operation there take place changes of the flow rate per second caused by the change in acceleration of the motion of the rocket, the altitude of levels of liquids in tanks, and so forth. Therefore, the calculation of the mass of the rocket in general should be produced by the formula (3.3).

By acceleration of terrestrial gravity g we mean pure Newton acceleration, caused by only the action of the force of mutual attraction between earth and the rocket. Since earth is considered a sphere, the acceleration of terrestrial gravity depends only on the distance of the point to the center of the earth:

$$g = \frac{fM}{r^2} = \frac{g_0 R^2}{r^2} \tag{3.4}$$

and is directed to the center of the earth.

Here $f = 6.670 \cdot 10^{-11}$ m^3/kg$\cdot$s^2 is the gravitational constant; $M = 5.9763 \cdot 10^{24}$ kg is the mass of earth ($fM = 3.9862 \times 10^{14}$ m^3/s^2); $g_0 = 9.8204$ m/s^2 is the acceleration of terrestrial gravity at the surface of earth.

Usually the acceleration of terrestrial gravity is united into one quantity with centrifugal acceleration caused by the rotation of earth, since the physical manifestation of both accelerations for bodies quiescent on the surface of earth is absolutely equal. But we will not do this in general because the magnitude of these accelerations is determined by various factors.

§ 4. Atmosphere

Terrestrial atmosphere is the medium in which flight of the rocket occurs. For a determination of the quantity of forces having an effect on the rocket it is necessary to know the basic characteristics of this medium: density, pressure and temperature. These quantities greatly depend on a number of factors: altitude of the point above the surface of the earth, geographic latitude, time of season and day, and so forth. But for practical purposes there is taken into account the dependence of characteristics of the atmosphere only on altitude. This dependence is given in tables of standard atmosphere [4] utilized during calculations of trajectories. The atmosphere is considered motionless, i.e., wind is not considered.

Temperature T, pressure p and air density ρ are connected with each other and with the altitude above the surface of the earth by the equation of state

$$p = \rho RT \tag{4.1}$$

and by the differential relation of equilibrium

$$dp = -g\rho\,dh. \tag{4.2}$$

Here $R = 287.05 \frac{m^2}{s^2 \cdot deg}$ is the gas constant for 1 kg of mass of air.

Excluding ρ from (4.1) and (4.2), we will obtain

$$\frac{dp}{p} = -\frac{g\,dh}{RT}$$

and after integration from p_0 to p and from 0 to h:

$$\ln\frac{p}{p_0} = -\int_0^h \frac{g\,dh}{RT}.$$

or

$$\frac{p}{p_0}=e^{-\int_0^h \frac{g\,dh}{RT}}. \qquad (4.3)$$

From expression (4.1) it follows:

$$\frac{\rho}{\rho_0}=\frac{T_0}{T}\frac{p}{p_0}$$

or, inserting p/p_0 from (4.3),

$$\frac{\rho}{\rho_0}=\frac{T_0}{T}e^{-\int_0^h \frac{g\,dh}{RT}}. \qquad (4.4)$$

§ 5. Aerodynamic Forces

Flight with nonoperating engine. Aerodynamic forces are the result of the influence of the environment on the surface of the rocket during its motion. From the general surface of the rocket S we separate the external surface of the body S_e, and the surface, more accurately the section of the exit plane of the nozzle S_a. The surface of jet vanes and forces having an effect on it are not as yet examined. Acting on every element of the surface are, in general, the normal force σdS and tangent force τdS (σ and τ, consequently, designated the normal and tangent forces having an effect per unit of surface area of the rocket at the examined point; see Fig. 5.1). The total force having an effect per unit of surface area of the rocket will be designated by p, so that

$$p=\sigma+\tau. \qquad (5.1)$$

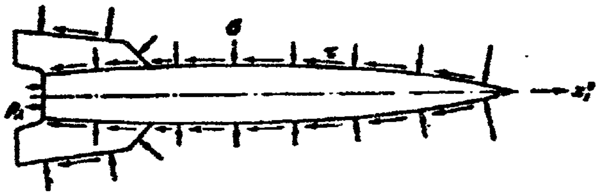

Fig. 5.1.

If rocket is motionless, then $\tau = 0$, and $\sigma = p$ (p — air pressure). During motion of the rocket $\tau \neq 0$ and $\sigma \neq p$. The difference

$$\sigma'=\sigma-p \qquad (5.2)$$

is the excess pressure of air on the surface of the rocket. It can be positive and negative. In the latter case it is called also rarefaction, which is created at a given point of the surface of the rocket.

The force R, appearing as a result of the influence of air on the whole surface of the rocket, is equal to

$$R=\int_S p\,dS. \qquad (5.3)$$

This integral can be divided into two integrals, the external surface of the body of the rocket S_e and the area of the nozzle exit section S_a:

$$R=\int\limits_{S_e} p\,dS+\int\limits_{S_a} p\,dS. \tag{5.4}$$

For the motionless rocket the pressure of air over the entire surface of the rocket is balanced:

$$R=0, \tag{5.5}$$

but each of the two integrals

$$\int\limits_{S_e} p\,dS \quad \text{and} \quad \int\limits_{S_a} p\,dS \tag{5.6}$$

is not equal to zero. Let us designate the vector with length p, directed on the normal to the element of the surface dS, by p_H.

Integrals (5.6) for the motionless rocket can be recorded in the form

$$\int\limits_{S_e} p_{\text{н}}\,dS \quad \text{and} \quad \int\limits_{S_a} p_{\text{н}}\,dS,$$

since p in this case coincides with p_H, and equality (5.5) takes the form

$$R=\int\limits_{S_e} p_{\text{н}}\,dS+\int\limits_{S_a} p_{\text{н}}\,dS=0. \tag{5.7}$$

Inasmuch as surface S_a is flat and perpendicular to the axis O_1x_1, integral $\int\limits_{S_a} p_{\text{н}}\,dS$, which is the force of air pressure on this surface, is equal to

$$\int\limits_{S_a} p_{\text{н}}\,dS=S_a p x_1^0, \tag{5.8}$$

and on the basis of (5.7)

$$\int\limits_{S_e} p_{\text{н}}\,dS=-\int\limits_{S_a} p_{\text{н}}\,dS=-S_a p x_1^0. \tag{5.9}$$

The integral of the form $\int\limits_{S} p_{\text{н}}\,dS$ is called aerostatic force having an effect on the surface S. Equality (5.7) shows that the aerostatic force, having an effect on the whole surface of the rocket, is equal to zero. Strictly speaking, this force, according to the law of Archimedes, is equal to the weight of air in the volume occupied by the rocket. But this quantity can be fully disregarded because of its smallness in comparison with not only the remaining forces having effect on the rocket, but also with errors of determination of these forces. Formula (5.9) gives the magnitude of the aerostatic force having an effect on the external surface of the body of the rocket.

Returning to the case of the moving rocket, each of the integrals entering into formula (5.4), on the basis of expression (5.1), will be divided into the sum of two integrals

$$R = \int_{S_e} \sigma\, dS + \int_{S_e} \tau\, dS + \int_{S_a} \sigma\, dS + \int_{S_a} \tau\, dS.$$

In this expression each of the integrals from the normal force $\sigma = p_H + \sigma'$ (where σ' is the excess normal pressure) in turn can be represented in the form of the sum of the two integrals:

$$R = \int_{S_e} p_H\, dS + \int_{S_e} \sigma'\, dS + \int_{S_e} \tau\, dS + \int_{S_a} p_H\, dS + \int_{S_a} \sigma'\, dS + \int_{S_a} \tau\, dS. \qquad (5.10)$$

The sum of the first and fourth terms in this expression (see formula (5.7)) is equal to zero.

Forces determined by integrals of the form $\int_S \sigma'\, dS$ and $\int_S \tau\, dS.$ and also by sums of such integrals bear the name of aerodynamic forces. Aerodynamic forces turn into zero both for the whole motionless rocket and for separate sections of its surface.

Equalities (5.10) and (5.7) show that the force R, with which air acts on the whole surface of the rocket with a nonoperating engine, constitutes an aerodynamic force which we will call full aerodynamic force.

The motion of air in the nozzle exit section can be disregarded and then the tangent forces will disappear:

$$\tau|_{S_a} = 0. \qquad (5.11)$$

(from this it follows that the sixth term in formula (5.10) turns into zero), and the normal pressure about the quantity will be constant:

$$\sigma'|_{S_a} = \mathrm{const.}$$

This constant will be designated $\sigma'_{\text{Д}}$, and let us call the bottom rarefaction for the nozzle section of the engine (it is assumed that the rocket does not fly with the nozzle forward and, consequently, $\sigma'_{\text{Д}} < 0$). The constant pressure $\sigma'_{\text{Д}}$ on the surface S_a gives a force of pressure $X_{1\text{Д}}$, equal to

$$X_{1\text{Д}} = \int_{S_a} \sigma'\, dS = S_a \sigma'_{\text{Д}} x_1^0. \qquad (5.12)$$

i.e., the fifth term in formula (5.10) constitutes a force in magnitude equal to $S_a |\sigma'_{\text{Д}}|$ and directed along the axis of the rocket from the summit to the tail.

Force $X_{1\text{Д}}$ will be called base drag, or the suction drag behind the engine nozzle.

Let us note that the base drag will be formed not only behind the nozzle but also behind other face areas on the rocket which we have included in the external surface S_e. Base drag $X_{1\text{Д}e}$, forming behind these areas, will . a part of the integral $\int_{S_e} \sigma'\, dS.$. This whole integral is the resultant of excess pressures about the external surface of the rocket body. Let us expand this resultant into two vectors: vector X_{1B}, directed along the axis of the rocket, and vector Y_1, directed along the perpendicular to the axis:

$$\int\limits_{S_e} \sigma' \, dS = X_{1B} + Y_1. \qquad (5.13)$$

We will call force X_{1B} the axial force of pressure, and force Y_1 the normal or lateral aerodynamic force.

Finally, the third term in expression (5.10) constitutes the resultant of tangents forces, or forces of friction, about the external surface of the rocket. This resultant is almost exactly directed along the longitudinal axis of the rocket. We will disregard its deflection from the longitudinal axis of the rocket and consider only the axial component of this force, which we denote by X_{1Tp} and call the axial frictional force:

$$X_{1Tp} = \int\limits_{S_e} \tau \, dS. \qquad (5.14)$$

Thus the expression (5.10), on the basis of equalities (5.7), (5.11)-(5.14), can be thus recorded:

$$R = X_{1B} + X_{1Tp} + X_{1д} + Y_1. \qquad (5.15)$$

The sum of the first three terms (5.15) constitutes a force directed along the axis of the rocket, which we will call axial aerodynamic force and will designate by X_1:

$$X_1 = X_{1B} + X_{1Tp} + X_{1д}. \qquad (5.16)$$

Finally

$$R = X_1 + Y_1. \qquad (5.17)$$

If the axis of the rocket is directed along the tangent to the trajectory, then the flowing around of the rocket will be symmetric relative to its axis. The distribution of pressures and forces of friction will be symmetric and, consequently, the normal aerodynamic force will be equal to zero.

If, however, the axis of the rocket will form with the tangent to the trajectory a certain angle α, called the angle of attack, then for rockets close in form to the solid of revolution (and only such rockets are examined by us), the flowing around will be symmetric with respect to the plane passing through the axis of the rocket and through the tangent to the trajectory. With this the normal aerodynamic force and, consequently, the full aerodynamic force will be disposed in this plane.

During normal flight of the rocket the angle of attack occurs small, of the order of several degrees. Experimental and theoretical research shows that for such angles of attack the axial aerodynamic force and all its components depend little on the angle of attack, and the normal aerodynamic force is directly proportional to the angle of attack:

$$Y_1 = Y_1' \alpha. \qquad (5.18)$$

The full aerodynamic force is frequently distributed not on the axial and normal components but on the drag X directed along the tangent to the trajectory opposite the direction of the motion of the rocket and lift Y, directed along the normal to the trajectory (Fig. 5.2). On the figure point O_1 is the center of gravity of the rocket, and point D is the center of pressure (point of application of force R).

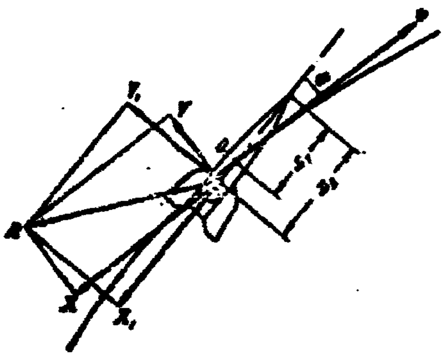

Fig. 5.2.

Let us find the expression for the value of drag and lift. Projecting forces X_1 and Y_1 on directions of the tangent and normal to the trajectory, we obtain

$$\left.\begin{aligned} X &= X_1\cos\alpha + Y_1\sin\alpha,\\ Y &= -X_1\sin\alpha + Y_1\cos\alpha. \end{aligned}\right\} \qquad (5.19)$$

For small angles of attack one can assume that $\cos\alpha = 1$, and $\sin\alpha = \alpha$; using equality (5.18), we copy these expressions in the form

$$\left.\begin{aligned} X &= X_1 + Y_1\alpha = X_1 + Y_1'\alpha^2,\\ Y &= -X_1\alpha + Y_1 = (Y_1' - X_1)\alpha = Y'\alpha, \end{aligned}\right\} \qquad (5.20)$$

where

$$Y' = Y_1' - X_1. \qquad (5.21)$$

Solving equation (5.19) with respect to X_1 and Y_1, we find inverse relation:

$$X_1 = X\cos\alpha - Y\sin\alpha,$$
$$Y_1 = X\sin\alpha + Y\cos\alpha,$$

or, approximately,

$$\left.\begin{aligned} X_1 &= X - Y\alpha = X - Y'\alpha^2,\\ Y_1 &= X\alpha + Y = (X + Y')\alpha. \end{aligned}\right\} \qquad (5.22)$$

Usually aerodynamic forces are expressed thus

$$X = c_x qS, \qquad (5.23)$$
$$Y = c_y qS = c_y' qS\alpha, \qquad (5.24)$$
$$X_1 = c_{x_1} qS, \qquad (5.25)$$
$$Y_1 = c_{y_1} qS = c_{y_1}' qS\alpha, \qquad (5.26)$$

where $q = \rho \frac{v^2}{2}$ – velocity head; ρ – air density at a given point of the trajectory; S – characteristic area of the rocket (for example, area of the midsection – the largest cross section); c_x, c_y, c_y', c_{x_1}, c_{y_1}, c_{y_1}' – dimensionless coefficients bearing the name of aerodynamic coefficients.

Powered flight. We will consider that the distribution of pressures and forces of friction about the external surface of the body of the rocket does not depend on the operation of the engine, i.e., during flight with an operating engine it remains the same as with a nonoperating engine. Regarding the nozzle exit section, the operation of the engine excludes any influence of environment on the plane of this section.

Designating the average pressure in the nozzle exit section by p_a, we can write

$$\int_{S_a} p\,dS = S_a p_a x_1^0. \qquad (5.27)$$

With the operation of the engine the pressure p_a and also the whole integral depend only on operating conditions of the engine, and action of air on the rocket appears only in the form of an integral on the external surface of the body of the rocket

$$\int_{S_e} p\,dS. \tag{5.28}$$

Force p, having an effect per unit of surface area of the rocket, as before can be represented in the form of the sum of normal atmospheric pressure $p_{\text{н}}$, excess pressure σ' and frictional force τ. Accordingly the integral (5.28) can be recorded in the form of the sum of the integrals

$$\int_{S_e} p\,dS = \int_{S_e} p_{\text{н}}\,dS + \int_{S_e} \sigma'\,dS + \int_{S_e} \tau\,dS. \tag{5.29}$$

For each of these integrals former expressions and designations (5.9), (5.13) and (5.14) remain in force, and therefore

$$\int_{S_e} p\,dS = -S_a p x_1^0 + X_{1\text{в}} + Y_1 + X_{1\text{тр}}. \tag{5.30}$$

In the case of the motionless rocket $p = p_{\text{н}}$, $\sigma' = 0$, $\tau = 0$, and acting on the rocket from the side of the external surface is only the aerostatic force

$$\int_{S_e} p_{\text{н}}\,dS = -S_a p x_1^0. \tag{5.31}$$

This force will be united with integral (5.27), and we will call the sum

$$P_{\text{ст}} = \int_{S_a} p\,dS + \int_{S_e} p_{\text{н}}\,dS = S_a(p_a - p)x_1^0 \tag{5.32}$$

static thrust.

Let us call the full aerodynamic force for the rocket with an operating engine the sum of only aerodynamic forces in the formula (5.30)

$$R_{\text{р.д}} = X_{1\text{в}} + X_{1\text{тр}} + Y_1. \tag{5.33}$$

Thus for the rocket with an operating engine the external surface force is composed of the aerodynamic force $R_{\text{р.д}}$ and static thrust $P_{\text{ст}}$. Comparing (5.33) with (5.15), we see that into the full aerodynamic force with the operating engine there does not enter drag of suction behind the nozzle of the engine, and in other respects it coincides with the full aerodynamic force with a nonoperating engine:

$$R = R_{\text{р.д}} + X_{1\text{д}}. \tag{5.34}$$

Normal aerodynamic forces with an operating and nonoperating engine coincide, and the axial forces differ from each other by a value of the base drag behind the nozzle:

$$Y_{1\text{р.з}} = Y_1, \quad X_{1\text{р.з}} = X_{1\text{а}} + X_{1\text{тр}} = X_1 - X_{1\text{д}}. \tag{5.35}$$

Formulas of transition (5.20) from the axial and normal aerodynamic forces to drag and lift and also expressions (5.23)-(5.26) for these forces preserve their form for the case of powered flight.

§ 6. Control and Control Forces

Control system should hold in the assigned limits deviations of parameters of the motion of the rocket from their computed values and thereby provide the assigned accuracy of firing.

The complex of parameters measured and controlled by the control system can be rather diverse. Let us consider the simplest control system regulating only the angular parameters of the motion of the rocket around the center of gravity.

The control system should consist of sensing devices which react to deflections of the rocket from the assigned law of motion and measure these deflections, effectors which create forces necessary for the change in motion of the rocket, and converting devices which receive signals from the sensing devices and produce commands for the effectors.

Since the rocket in motion with respect to the center of gravity possesses three degrees of freedom (for us, in particular, these three degrees of freedom correspond to the three angles φ, ξ, η), the effectors of the control system should also have three degrees of freedom. With a smaller quantity of degrees of freedom of controls the latter cannot determine the motion of the rocket around the center of gravity by all three degrees of freedom; with a large quantity the problem of control becomes indefinite, since the assigned motion of rocket in this case will correspond to not one definite law of the motion of controls but an infinite number of such laws. But also under the condition that the number of degrees of freedom of controls is equal to three, there exists an unlimited possibility of the concrete realization of these organs.

In exactly the same way sensing devices of the control system can be fulfilled by the most diverse principles and in different form.

Consequently, equations connecting the motion of controls with the motion of the rocket (so-called control equations) can have an absolutely different form with various principles of operation and design of control system. In general they can be thus written:

$$\left.\begin{aligned} F_1[\delta_1(t), x(t), y(t), z(t), \varphi(t), \xi(t), \eta(t)] &= 0, \\ F_2[\delta_2(t), x(t), y(t), z(t), \varphi(t), \xi(t), \eta(t)] &= 0, \\ F_3[\delta_3(t), x(t), y(t), z(t), \varphi(t), \xi(t), \eta(t)] &= 0, \end{aligned}\right\} \tag{6.1}$$

where δ_1, δ_2, δ_3 are deflections of effectors of the control system; F_1, F_2, F_3 are functionals from functions taken in brackets, i.e., quantities depending not only on current values of these functions but also on their preceding values starting from the moment of launch. This dependence can be rather complicated.

In the subsequent account as an example we will dwell on the control system of the rocket the sensing devices of which are gyroscopic instruments — gyro horizon and vertical scope, effectors — jet vanes, and converting devices — amplifier-converter and control actuators.

The right and left jet vanes (see below Fig. 7.1) are deflected synchronqusly (and at an identical angle), and thus the number of degrees of freedom of controls is indeed equal to three.

Gyroscopic instruments each consist of a gyroscope and two frames, internal and external, the location of which at the time of launch is shown on Fig. 6.1.

In the flight the axes of rotation of the gyroscopes maintain a constant attitude, i.e., the axis of rotation of the gyroscope of the gyro horizon $O_r x_r$ remains all the time parallel to the axis Ox of the terrestrial system of coordinates, and the axis of rotation of the gyroscope of the vertical gyro $O_B z_B$ remains parallel to the axis Oz of the terrestrial coordinate system. Axes of rotation of external frames of the gyro-instruments are connected with the body of the rocket. Consequently, in flight the axis of rotation of the external frame of the gyro horizon $O_r z_r$ is parallel to axis $O_1 z_1$ of the bound system of coordinates, and the axis of rotation of the external frame of the vertical gyro $O_B x_B$ is parallel to axis $O_1 y_1$ of the bound system.

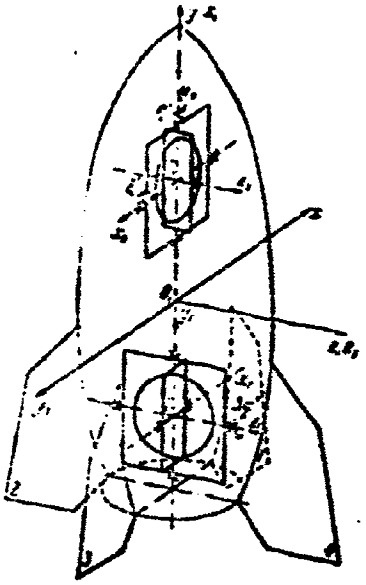

Fig. 6.1.

We will characterize directions of axes by their unit vectors so that

$$x_{г}^{0}=x^{0}, \tag{6.2}$$
$$z_{в}^{0}=z^{0}, \tag{6.3}$$
$$z_{г}^{0}=z_{1}^{0}, \tag{6.4}$$
$$x_{в}^{0}=y_{1}^{0}. \tag{6.5}$$

Now it is easy to determine the direction of the intermediate axes of gyroscopic instruments — axes of rotation of internal frames. The intermediate axis of the gyro horizon $O_r y_r$ is always perpendicular to two other axes of this instrument, $O_r x_r$ and $O_r z_r$, and consequently,

$$y_{г}^{0}=\frac{z_{г}^{0}\times x_{г}^{0}}{|z_{г}^{0}\times x_{г}^{0}|}=\frac{z_{1}^{0}\times x^{0}}{|z_{1}^{0}\times x^{0}|}. \tag{6.6}$$

In exactly the same way the intermediate axis of the vertical gyro $O_B y_B$ is perpendicular to $O_B x_B$ and $O_B z_B$ and, therefore

$$y_{в}^{0}=\frac{x_{в}^{0}\times z_{в}^{0}}{|x_{в}^{0}\times z_{в}^{0}|}=\frac{y_{1}^{0}\times z^{0}}{|y_{1}^{0}\times z^{0}|}. \tag{6.7}$$

With the help of Table 2.1 we obtain

$$\begin{aligned} z_{1}^{0}\times x^{0}=z_{1}^{0}\times\left(x_{1}^{0}\cos\varphi-y_{1}^{0}\sin\varphi+z_{1}^{0}(\xi\cos\varphi+\eta\sin\varphi)\right)&= \\ =y_{1}^{0}\cos\varphi+x_{1}^{0}\sin\varphi&, \\ y_{1}^{0}\times z^{0}=y_{1}^{0}\times(-x_{1}^{0}\xi+y_{1}^{0}\eta+z_{1}^{0})=z_{1}^{0}\xi+x_{1}^{0}&. \end{aligned}$$

As before, disregarding the second degrees of quantities ξ and η, we obtain that

$$|z_{1}^{0}\times x^{0}|=|y_{1}^{0}\times z^{0}|=1$$

and, consequently, formulas (6.6) and (6.7) can be rewritten in the form

$$y_{г}^{0}=x_{1}^{0}\sin\varphi+y_{1}^{0}\cos\varphi, \tag{6.8}$$
$$y_{в}^{0}=x_{1}^{0}+z_{1}^{0}\xi. \tag{6.9}$$

If in flight the axis of the rocket has an assigned direction, i.e., the angles ξ and η are equal to zero and angle φ is equal to the program angle $\varphi_{пр}$, then cursors of the potentiometers are at a zero position. In the vertical gyro this is attained automatically, since the entire instrument, together with the body of the rocket, turns about the axis of the spin of the rotor, and the relative position of the frames does not change. In gyro horizon both frames and the cursor of the potentiometer connected with the external frame do not change position relative to the earth's axis, and so that the cursor will remain at zero the body of the potentiometer turns about the axis O_1z_1 with respect to the body of the rocket at the same angle at which the rocket should be inclined according to the program from its initial position (i.e., at an angle of $90^\circ - \varphi_{\text{np}}$). With deflection of the rocket from the assigned position the angles ξ, η and $\Delta\varphi = \varphi - \varphi_{\text{np}}$ become different from zero, and there simultaneously appear displacements of cursors of potentiometers from the zero position: ξ' — displacement (angle) of the potentiometer cursor on the axis of rotation of the external frame of the vertical gyro, η' — displacement of the cursor of the potentiometer on the intermediate axis of the vertical gyro, and $\Delta\varphi$ — displacement of the potentiometer cursor of the gyro horizon (on the axis of rotation of the external frame).

Examining the diagram of gyro-instruments (Fig. 6.1), it is easy to check that with the appropriate selection of directions of the reading the angle $90^\circ - \xi'$ is equal to the angle between axes O_1z_1 and O_By_B, angle $90^\circ - \eta'$ is equal to the angle between axes O_1y_1 (O_Bx_B) and O_Bz_B, and angle $\varphi' = \varphi_{\text{np}} + \Delta\varphi'$ is equal to the angle between axes O_1y_1 and O_ry_r. The connection between angles ξ, η, $\Delta\varphi$ and ξ', η', and $\Delta\varphi'$ can be obtained by calculating scalar products of unit vectors of corresponding axes with the help of formulas (6.9), (6.3) and (6.8):

$$\begin{gathered}\cos(90^\circ - \xi') = z_1^0 \cdot y_B^0 = z_1^0 \cdot (x_1^0 + z_1^0\xi),\\ \cos(90^\circ - \eta') = y_1^0 \cdot z_B^0 = y_1^0 \cdot (-x_1^0\xi + y_1^0\eta + z_1^0),\\ \cos(\varphi_{np} + \Delta\varphi') = y_1^0 \cdot y_r^0 = y_1^0 \cdot (x_1^0 \sin\varphi + y_1^0 \cos\varphi),\end{gathered}$$

whence

$$\begin{gathered}\sin\xi' = \xi,\\ \sin\eta' = \eta,\\ \cos(\varphi_{np} + \Delta\varphi') = \cos\varphi.\end{gathered}$$

These relations permit being once again convinced of the smallness of angles ξ' and η', which enables replacing their sines by the angles themselves and writing following formula:

$$\xi' = \xi, \tag{6.10}$$

$$\eta' = \eta. \tag{6.11}$$

and also

$$\Delta\varphi' = \varphi - \varphi_{np} = \Delta\varphi. \tag{6.12}$$

Formulas (6.10)-(6.12) give the connection between deflections of the rocket from the assigned position and reaction of gyro-instruments on these deflections.

We will not touch upon the work of the amplifier-converter and control actuators, which convert the voltages taken from potentiometers of the gyro-instruments, which are directly proportional to displacements of cursors of these potentiometers, into angles of deflection of the jet vanes. Let us consider the concluding link of the control circuit — forces having an effect on the control surfaces.

The full force having an effect on the control surfaces found in the gas flow will be decomposed into three components — drag of the control surface Q_p directed along the axis of flow, i.e., along the axis of the rocket, lift R_p directed perpendicular to the axis of the rocket and to the axis of the control surface, and axial force T_p directed in parallel to the axis of the control surface. The last force is small, and therefore we will disregard it, especially as for two opposite control surfaces the axial forces are balanced (wholly, if the angles of deflection of the control surfaces are identical, and partially if the angles of deflection are different).

Approximately one can assume that the lift of the control surface is proportional to the angle of deflection of the control surface:

$$R_p = R'\delta, \tag{6.13}$$

and the drag of the control surface depends on the angle of deflection of the control surface according to the parabolic law:

$$Q_p = Q_{p0} + \lambda\delta^2. \tag{6.14}$$

Furthermore, forces having an effect on the jet vane can be expressed as any gas-dynamic forces in the form

$$\left.\begin{aligned} Q_p &= c_Q \frac{\rho_p u_p^2}{2} S_p, \\ R_p &= c_R \frac{\rho_p u_p^2}{2} S_p = c_R' \frac{\rho_p u_p^2}{2} S_p \delta, \end{aligned}\right\} \tag{6.15}$$

where ρ_p — density of gas in the section of the stream of the engine passing through the leading edge of the control surface; u_p — speed of gas flow in the same section; S_p — area of control surface; c_Q, c_R, c_R' — gas-dynamic coefficient depending on the Mach number of gas flow and on the angle of deflection of the control surface.

In the first approximation c_Q, analogous to the drag of the control wheel Q_p, depends on the angle of deflection of the control surface by the parabolic law, c_R is proportional to the angle of deflection of the control surface, and c_R' thus does not depend on this angle.

Subsequently we will examine only the total forces for all four control surfaces: axial force X_{1p}, equal to the sum of drags of the four control surfaces,

$$X_{1p} = Q_{p1} + Q_{p2} + Q_{p3} + Q_{p4},$$

or, on the basis of (6.14),

$$X_{1p} = 4Q_{p0} + \lambda(\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2). \tag{6.16}$$

and lateral forces: Y_{1p}, equal to the sum of lifts of control surfaces 2 and 4, and Z_{1p}, equal to the sum of lifts of control surfaces 1 and 3.

Directions of reading of angles of deflection of control surfaces will be selected in such a manner that the positive angles of deflection of the control surfaces correspond to positive lateral forces. T[illegible] we will consider positive the deflections of control surfaces 2 and 4 downwards, [illegible]nd control surfaces 1 and 3 – to the left, depending on the flight (Fig. 7.1). [illegible]h such condition we obtain

$$\left.\begin{aligned} Y_{1p} &= R'\delta_2 + R'\delta_4 = 2R'\delta_2, \\ Z_{1p} &= R'\delta_1 + R'\delta_3. \end{aligned}\right\} \qquad (6.17)$$

§ 7. Moments of Forces

Let us find expressions for moments of forces examined by us with respect to the center of gravity of the rocket. We will consider that the center of gravity lies on the axis of the rocket at a distance x_T from the summit.

Gravity G always acts along a straight line passing through the center of gravity and does not create a moment with respect to the center of gravity.

Up till now only the magnitude and direction of aerodynamic forces were discussed. The line of action is fully determined only for the complete aerodynamic force R. The point of intersection of this line of action with the axis of the rocket is called the center of pressure. Let us agree to consider the full aerodynamic force applied in the center of pressure; then the lines of action of all components of this force X_1, Y_1, X, Y and so forth will pass through the center of pressure (see Fig. 5.2).

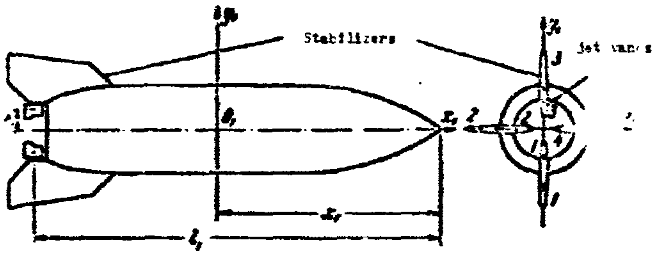

Fig. 7.1.

Thus the axial aerodynamic force X_1 acts along the axis of the rocket and therefore does not create a moment with respect to the center of gravity. The same can be said in the case of powered flight for static thrust P_{CT}.

The normal aerodynamic force Y_1 creates with respect to the center of ravity of the rocket the moment M_a, equal in value to

$$M_a = Y_1(x_Д - x_T). \qquad (7.1)$$

where $x_Д$ is the distance from the summit of the rocket to the center of pressure.

This moment, just as the full aerodynamic force, acts in a plane passing through the axis of the rocket and through the tangent to the trajectory; in other words, the vector of this moment is perpendicular to the axis of the rocket and to the tangent to the trajectory.

If the center of pressure is behind the center of gravity, then the moment of normal aerodynamic force acts on the decrease of the angle of attack and is called in this case the stabilizing aerodynamic moment, and the rocket with such location of the center of pressure and center of gravity is statically stable. If, however, the center of pressure lies ahead of the center of gravity, then the rocket is called statically unstable; the moment of the normal aerodynamic force acts for such rocket on the increase of the angle of attack and bears the name destabilizing aerodynamic moment.

Let us consider moments of forces from jet vanes.

The point of intersection of the line of action of the full gas-dynamic force having an effect on the control surface with the plane of symmetry of the control surface will be called the center of pressure of the control surface. The position of the center of pressure is changed during movement of the control surface, but this change will be disregarded, is considering that the center of pressure of the control surface always lies in the plane of the corresponding stabilizer at a distance l_1 from the summit of the rocket and at distance h_1 from the axis of the rocket (Fig. 7.1).

Drag of the control surface Q_p creates with respect to the center of gravity of the rocket a moment equal in value to

$$M_{Q_p} = Q_p h_1. \tag{7.2}$$

For control surfaces 2 and 4 force of Q_p are equal with each other, since the angles of deflection of the control surfaces are identical and, consequently, moments M_{Q_p} are equal in value. But the directions of these moments are opposite, and therefore they are mutually balanced. Angles of deflection of control surfaces 1 and 3, in general, can be different, but this difference at small angles of deflection very little affects the vane drag and, consequently, the moments M_{Q_p}; these moments will also be disregarded.

Lift of the control surface R_p creates with respect to the center of gravity of the rocket a moment which can be represented in the form of the sum of two moments with respect to two mutually perpendicular axes passing through the center of gravity: the moment with respect to the longitudinal axis of the rocket equal in value to

$$M_{x_1} = R_p h_1. \tag{7.3}$$

and the moment with respect to the lateral axis parallel to the axis of rotation of the control surface

$$M_{y_1(z_1)} = R_p (l_1 - x_T). \tag{7.4}$$

We will find the total moments from jet vanes relative to the bound axes if we substitute in (7.3) and (7.4) expressions (6.13) for lift of the control surfaces and consider rule signs for angles of deflection of control surfaces:

$$\left.\begin{aligned} M_{x_1} &= R' h_1 (-\delta_1 + \delta_2 + \delta_3 - \delta_4) = R' h_1 (\delta_3 - \delta_1). \\ M_{y_1} &= R' (l_1 - x_T)(\delta_1 + \delta_3). \\ M_{z_1} &= R' (l_1 - x_T)(-\delta_2 - \delta_4) = -2R' (l_1 - x_T)\delta_2. \end{aligned}\right\} \tag{7.5}$$

§ 8. Damping Moments

Up till now, in examining aerodynamic forces having an effect on the rocket, we were not interested in its angular velocity. Strictly speaking, our certain affirmations and formulas, for example (7.1), are true only at an angular velocity of the rocket equal to zero. If the rocket flies with speed v and angle of attack α and has at this angular velocity $\omega \neq 0$, then the flowing around of the rocket and distribution of pressures along its surface will be different than when $\omega = 0$.

Consequently, aerodynamic forces and moments depend not only on v, α, ρ, T, but also on ω. There is importance not only of the value of angular velocity but also its direction with respect to the axes connected with the rocket.

For an approximate appraisal of the magnitude of additional forces and moments appearing with the rolling of the rocket, let us examine the motion of the rocket in plane $O_1x_1y_1$, being limited for simplicity by the case of the zero angle of attack.

If the speed of the center of gravity of the rocket is equal to v, and the component of angular velocity along the axis O_1z_1 perpendicular to the examined plane is equal to ω_{z_1}, then the point of the body of the rocket found at distance x_1 from the center of gravity of the rocket, has, besides speed v, a peripheral velocity $\omega_{z_1}x_1$. Consequently, the full speed of this point will form with the speed of the center of gravity the angle

$$\Delta\alpha=\frac{\omega_{z_1}x_1}{v}. \tag{8.1}$$

By this magnitude is changed the angle of attack in the examined point of the body.

These additional angles of attack are the cause of the appearance of additional forces and moments. The mean value of the additional angle of attack on the stabilizer is equal to

$$\Delta\alpha_{\text{ст}}=\frac{\omega_{z_1}(x_{\text{д.ст}}-x_{\text{т}})}{v}, \tag{8.2}$$

where $x_{\text{д.ст}}$ is the distance of the center of pressure of the stabilizer from the summit of the rocket.

The additional lift corresponding to this angle is equal to

$$\Delta Y_{\text{ст}}=c'_{y\,\text{ст}}qS\,\Delta\alpha_{\text{ст}}, \tag{8.3}$$

where

$$c'_{y\,\text{ст}}=\frac{\partial c_{y\,\text{ст}}}{\partial\alpha}=\frac{c_{y\,\text{ст}}}{\alpha}.$$

$c_{y\text{ст}}$ is the coefficient of lift of two blades of the stabilizer referred to the area S.

Having substituted (8.2) in (8.3), we obtain

$$\Delta Y_{\text{ст}}=c'_{y\,\text{ст}}\frac{\rho v^2}{2}S\frac{\omega_{z_1}(x_{\text{д.ст}}-x_{\text{т}})}{v}=\frac{1}{2}c'_{y\,\text{ст}}\rho S(x_{\text{д.ст}}-x_{\text{т}})v\omega_{z_1}. \tag{8.4}$$

This force is directed to the side opposite the direction of the motion of the stabilizer in the rotation of the rocket. It creates an additional moment effective in the direction opposite the direction of rotation of the rocket and is called therefore the damping moment. The magnitude of the damping moment will be

$$\Delta M_{z_1}=-\Delta Y_{\text{ст}}(x_{\text{д.ст}}-x_{\text{т}})=-\frac{1}{2}c'_{y\,\text{ст}}\rho S(x_{\text{д.ст}}-x_{\text{т}})^2v\omega_{z_1}. \tag{8.5}$$

In reality the magnitude of the damping moment is somewhat greater than that calculated by the formula (8.5), since the damping moment is created not only by the stabilizer but also the body. Therefore, not increasing the order of the error, it is possible to assume

$$c'_{y\,\text{ст}}=c'_y,\quad x_{\text{д.ст}}-x_{\text{т}}=\frac{l}{2}.$$

Then we will obtain the following approximate expressions with an error to the greater side:

$$\Delta Y_1 = \frac{1}{4} c'_y \rho S l v \omega_{z_1}. \tag{8.6}$$

$$\Delta M_{z_1} = -\frac{1}{8} c'_y \rho S l^2 v \omega_{z_1}. \tag{8.7}$$

Analogous expressions can be written for an estimate of moments with respect to the other lateral axis:

$$\Delta Z_1 = -\frac{1}{4} c'_y \rho S l v \omega_{y_1}, \tag{8.8}$$

$$\Delta M_{y_1} = -\frac{1}{8} c'_y \rho S l^2 v \omega_{y_1}. \tag{8.9}$$

Let us consider the rotation of the rocket about the longitudinal axis with an angular velocity ω_{x_1}. With such rotation the section of the stabilizer, which is at distance h from the axis of the rocket, has, besides speed v, the peripheral velocity $h\omega_{x_1}$. Consequently, the full speed of this section will form with the speed of the center of gravity the angle

$$\Delta\alpha = \frac{h\omega_{x_1}}{v}, \tag{8.10}$$

which also constitutes an increase in the angle of attack of the section of the stabilizer. On the average for the stabilizer, the increase in angle of attack from the rotation about the longitudinal axis consists of

$$\Delta\alpha_{\text{ст}} = \frac{h_{\text{ст}}\omega_{x_1}}{v},$$

where h_{cT} is the distance of the center of pressure of the stabilizer from the longitudinal axis.

This additional angle of attack causes an additional lift of the stabilizer blade equal to

$$\Delta Y = \frac{c'_{y\,\text{ст}}}{2} qS\Delta\alpha_{\text{ст}} = \frac{c'_{y\,\text{ст}}}{2}\frac{\rho v^2}{2} S \frac{\omega_{x_1} h_{\text{ст}}}{v}$$

(we designate the coefficient $\frac{c'_{ycT}}{2}$, since the lift of one blade is examined, and it is natural to refer coefficient c'_{ycT} to two blades), or

$$\Delta Y = \frac{1}{4} c'_{y\,\text{ст}} \rho S h_{\text{ст}} v \omega_{x_1}.$$

Force ΔY is directed opposite the direction of motion of the blade with rotation. Consequently, for two opposite blades forces ΔY will form a pair with the moment

$$-2\Delta Y_{\text{ст}} h_{\text{ст}} = -\frac{1}{2} c'_{y\,\text{ст}} \rho S h^2_{\text{ст}} v \omega_{x_1}.$$

The additional moment from all four blades is equal to

$$\Delta M_{x_1} = -c'_{y\,\text{ст}} \rho S h^2_{\text{ст}} v \omega_{x_1}.$$

Approximately one can assume that

$$\Delta M_{x_1} = -\frac{1}{4} c_y' \rho S l_{cT}^2 v \omega_{x_1}, \qquad (8.11)$$

where l_{cT} is the span of the stabilizer. This moment acts opposite the rotation of the rocket, i.e., it is also damping.

In the common form we will use the following expressions for damping moments:

$$\left.\begin{aligned} \Delta M_{x_1} &= -m_{x_1}^{\omega} S l^2 \rho v \omega_{x_1}, \\ \Delta M_{y_1} &= -m_{y_1}^{\omega} S l^2 \rho v \omega_{y_1}, \\ \Delta M_{z_1} &= -m_{z_1}^{\omega} S l^2 \rho v \omega_{z_1}, \end{aligned}\right\} \qquad (8.12)$$

where $m_{x_1}^{\omega}$ and $m_{y_1}^{\omega} = m_{z_1}^{\omega}$ are dimensionless coefficients of aerodynamic damping determined either with the help of special aerodynamic experiments or by means of more accurate aerodynamic designs.

Additional forces of (8.6) and (8.8), in view of their smallness, will be disregarded.

CHAPTER III

GENERAL EQUATIONS OF MOTION

§ 9. Equations of Motion in Vector Form

A rocket with an operating engine continuously expends mass contained in it, and therefore for the final interval of time the laws of dynamics of a solid or system to it are directly inapplicable. But for an infinitesimal interval of time dt it is possible to solve the problem of the motion of a rocket with the help of theorems of dynamics of the system with the following assumptions:

1. We examine as a single system all the masses contained in the rocket at the moment of time **t**.

2. Let us disregard forces having an effect on masses expending during time dt (i.e., from the moment t up to moment (**t** + dt) through the nozzle exit section.

We will call the mass exiting for time dt through the nozzle exit section infinitesimal waste material.

Thus the examined system coincides at the time **t** with the rocket and at the time **t** + d**t** consists of the rocket and waste material, and this is the first assumption. According to the second assumption, forces, having an effect on the examined system coincide with forces having an effect on the rocket.

Equations of motion of the rocket are easily obtained, proceeding from equations of the motion of the system, which, as is known, have such form:

$$\frac{d\mathbf{K}_c}{dt} = \mathbf{F}, \tag{9.1}$$

$$\frac{d\mathbf{L}_c^{(c)}}{dt} = \mathbf{M}^{(c)}, \tag{9.2}$$

where K_c — the momentum of the system; $\mathbf{F}$ — resultant (main vector) of external forces having an effect on the system; $\mathbf{L}_c^{(c)}$ — angular momentum of the system with respect to the center of gravity of the system; $\mathbf{M}^{(c)}$ — total moment (main moment) of external forces with respect to the center of gravity of the system.

In order to turn to equations of motion of the rocket, let us find first of all the expression for the momentum of the rocket. Let us divide the rocket into elementary particles, where for one of such particles we will take the infinitesimal waste product whose center of gravity coincides with the center of the exit section of the nozzle. The position of the center of gravity of the rocket at the moment of time t is determined by the equality

$$mr = \sum_\nu m_\nu r_\nu + \dot m\, dt\, r_a. \qquad (9.3)$$

where m – the mass of the rocket at the moment of time t; r – the radius vector of the center of gravity of the rocket relative to a certain motionless center; m_ν – the mass of the elementary particle not abandoning the rocket during the time dt; r_ν – the radius vector of the same particle; $\dot m$ – flow rate per second of mass (see § 3); $\dot m\, dt$ – the mass of waste material; r_a – the radius vector of the center of gravity of waste material, i.e., the geometric center of the nozzle exit section.

During the time dt the mass of the rocket is changed by the magnitude

$$dm = -\dot m\, dt, \qquad (9.4)$$

the radius vector of the center of gravity of the rocket by the magnitude

$$dr = v\, dt \qquad (9.5)$$

(v is the speed of the center of gravity of the rocket relative to the motionless system of coordinates – absolute velocity); the masses of particles are not changed, and their radii-vectors are changed by

$$dr_\nu = v_\nu dt. \qquad (9.6)$$

where v_ν is the absolute velocity of the particles. Waste material no longer enters into the rocket, and the position of its center of gravity is determined by the equality

$$(m - \dot m\, dt)(r + dr) = \sum_\nu m_\nu (r_\nu + dr_\nu). \qquad (9.7)$$

Subtracting expression (9.3) from (9.7), we will find

$$m\,dr - \dot m r\,dt - \dot m\,dr\,dt = \sum_\nu m_\nu dr_\nu - \dot m r_a dt. \qquad (9.8)$$

If in (9.8) we disregard the infinitesimal of higher order $\dot m\, dr\, dt$, and we replace dr and dr_ν by their expressions (9.5) and (9.6) and then reduce by dt we will then obtain

$$mv - \dot m r = \sum_\nu m_\nu v_\nu - \dot m r_a.$$

Noting that the sum $\sum_\nu m_\nu v_\nu$, correct to an infinitesimal momentum of waste material, coincides with the momentum of the rocket K, we can write the following expression for K;

$$K = \sum_\nu m_\nu v_\nu = mv + \dot m (r_a - r) = mv + \dot m b. \qquad (9.9)$$

where

$$b = r_a - r \qquad (9.10)$$

is the vector connecting the center of gravity of the rocket with the center of the nozzle exit section.

Differentiating (9.10), we find $\dot b = v_a - v$, i.e., the speed of the center exit section of the nozzle v_a is equal to

$$v_a = v + \dot b. \qquad (9.11)$$

Designating the speed of the center of gravity of waste material with respect to the center of the nozzle exit section (exit velocity) by $\mathbf{u}$, we will obtain for the absolute velocity of the center of gravity of waste material the expression

$$\mathbf{v}_{\text{отбр}} = \mathbf{v}_a + \mathbf{u} = \mathbf{v} + \dot{\mathbf{b}} + \mathbf{u}. \tag{9.12}$$

whence the momentum of waste material is equal to (since this value is infinitesimal, we designate it by $d\mathbf{K}_{\text{отбр}}$)

$$d\mathbf{K}_{\text{отбр}} = \dot{m}\, dt\, \mathbf{v}_{\text{отбр}} = \dot{m}(\mathbf{v} + \mathbf{u} + \dot{\mathbf{b}})\, dt. \tag{9.13}$$

At the time $\mathbf{t}$ the momentum of the examined system is equal to the momentum of the rocket $\mathbf{K}_c = \mathbf{K}$. At the moment of time $\mathbf{t} + dt$ the momentum of system is composed of the momentum of the rocket and infinitesimal momentum of waste material

$$\mathbf{K}_c + d\mathbf{K}_c = \mathbf{K} + d\mathbf{K} + d\mathbf{K}_{\text{отбр}}.$$

Consequently,

$$d\mathbf{K}_c = d\mathbf{K} + d\mathbf{K}_{\text{отбр}}. \tag{9.14}$$

The change in momentum of the rocket is easily found from (9.9):

$$d\mathbf{K} = m\, d\mathbf{v} + \mathbf{v}\, dm + \dot{m}\, d\mathbf{b} + \mathbf{b}\, d\dot{m} = m\, d\mathbf{v} - \mathbf{v}\dot{m}\, dt + \dot{m}\dot{\mathbf{b}}\, dt + \mathbf{b}\ddot{m}\, dt, \tag{9.15}$$

where

$$\ddot{m} = \frac{d\dot{m}}{dt} = -\frac{d^2 m}{dt^2}. \tag{9.16}$$

Inserting (9.13) and (9.15) in (9.14), we obtain

$$d\mathbf{K}_c = m\, d\mathbf{v} - \dot{m}\mathbf{v}\, dt + \dot{m}\dot{\mathbf{b}}\, dt + \ddot{m}\mathbf{b}\, dt + \dot{m}\mathbf{v}\, dt + \dot{m}\mathbf{u}\, dt + \dot{m}\dot{\mathbf{b}}\, dt = m\, d\mathbf{v} + \dot{m}\mathbf{u}\, dt + 2\dot{m}\dot{\mathbf{b}}\, dt + \ddot{m}\mathbf{b}\, dt. \tag{9.17}$$

Replacing $d\mathbf{K}_c$ in equation (9.1) by expression (9.17), we find the following vector equation of motion of the center of gravity of the rocket.

$$m\frac{d\mathbf{v}}{dt} + \dot{m}\mathbf{u} + 2\dot{m}\dot{\mathbf{b}} + \ddot{m}\mathbf{b} = \mathbf{F}. \tag{9.18}$$

Let us turn to the equation of motion of the rocket about the center of gravity.

At the moment of time $\mathbf{t}$ angular momentum of the system with respect to the center of gravity of the system $\mathbf{L}_c^{(c)}$ coincides with the angular momentum of the rocket relative to its center of gravity $\mathbf{L}$:

$$\mathbf{L}_c^{(c)} = \mathbf{L} = \sum_{\nu} (\mathbf{r}_\nu - \mathbf{r}) \times m_\nu \mathbf{v}_\nu + (\mathbf{r}_a - \mathbf{r}) \times \dot{m}\mathbf{v}_{\text{отбр}}\, dt. \tag{9.19}$$

Here, by examining the waste material as the elementary particle, we disregarded its intrinsic angular momentum.

At the moment of time $\mathbf{t} + dt$ the angular momentum of the system with respect to the center of gravity of the system is composed of the angular momentums of the rocket and waste material relative to this point:

$$\mathbf{L}_c^{(c)} + d\mathbf{L}_c^{(c)} = \mathbf{L}_{(t+dt)}^{(c)} + d\mathbf{L}_{\text{отбр}}^{(c)}. \tag{9.20}$$

Angular momentums in (9.20) are determined by the well-known formulas

$$\mathbf{L}_{(t+dt)}^{(c)} = \mathbf{L} + d\mathbf{L} + (\mathbf{r} + d\mathbf{r} - \mathbf{r}_c - d\mathbf{r}_c) \times (\mathbf{K} + d\mathbf{K}). \tag{9.21}$$

$$d\mathbf{L}_{\text{отбр}}^{(c)} = d\mathbf{L}_{\text{отбр}} + (\mathbf{r}_a + d\mathbf{r}_{\text{отбр}} - \mathbf{r}_c - d\mathbf{r}_c) \times d\mathbf{K}_{\text{отбр}}. \tag{9.22}$$

where $L + dL$ is the angular momentum of the rocket relative to its center of gravity and $dL_{\text{отбр}}$ is the intrinsic angular momentum of waste material with which, furthermore, we will disregard.

Radius vectors of centers of gravity of the system, rocket and waste material are connected by the relation

$$m(r_c + dr_c) = (m - \dot{m}\,dt)(r + dr) + \dot{m}\,dt\,(r_a + dr_{\text{отбр}}),$$

or, correct to infinitesimals of the second order,

$$r_c + dr_c = r + dr - \frac{\dot{m}}{m} r\,dt + \frac{\dot{m}}{m} r_a\,dt = r + dr + \frac{\dot{m}}{m} b\,dt. \tag{9.23}$$

We insert obtained expressions (9.21)-(9.23) in the relation (9.20):

$$L_c^{(c)} + dL_c^{(c)} = L + dL - \frac{\dot{m}}{m} b\,dt \times (K + dK) + \\ + \left(r_a + dr_{\text{отбр}} - r - dr - \frac{\dot{m}}{m} b\,dt\right) \times dK_{\text{отбр}}$$

and, comparing with the first part of equality (9.19), we find correct to infinitesimals of the second order:

$$dL_c^{(c)} = dL - \frac{\dot{m}}{m} b \times K\,dt + (r_a - r) \times dK_{\text{отбр}}. \tag{9.24}$$

Now using expressions (9.9) and (9.13) for K and $dK_{\text{отбр}}$, we obtain the following expression for $dL_c^{(c)}$:

$$dL_c^{(c)} = dL - \frac{\dot{m}}{m} b \times (mv + \dot{m}b)\,dt + b \times (v + u + \dot{b})\,\dot{m}\,dt = dL + \dot{m} b \times (u + \dot{b})\,dt. \tag{9.25}$$

We replace $dL_c^{(c)}$ in equation (9.2) by expression (9.25):

$$\frac{dL}{dt} + \dot{m} b \times (u + \dot{b}) = M^{(c)}.$$

Since at the moment of time t the rocket and system coincide, instead of $M^{(c)}$ it is possible to write the sum of moments of external forces with respect to the center of gravity of the rocket M; then we will obtain the equation of motion of the rocket with respect to the center of gravity in the form

$$\frac{dL}{dt} + \dot{m} b \times (u + \dot{b}) = M. \tag{9.26}$$

The derivative of vector b in the motionless system of coordinates can be thus represented:

$$\dot{b} = \frac{db}{dt} = \frac{\partial b}{dt} + \omega \times b. \tag{9.27}$$

where $\partial b/dt$ is the derivative of vector b with respect to the body of the rocket (local derivative) and ω is the angular velocity of rotation of the body of the rocket.

But we assume that the center of gravity of the rocket and center of the nozzle exit section lie on the longitudinal axis of the rocket, and therefore vector b, and, consequently, and its local derivative, will be parallel to the axis of the rocket. It follows from this that

$$b \times \frac{\partial b}{dt} = 0. \tag{9.28}$$

Disregarding geometric and gas-dynamic asymmetry of the expiration of gases, we will consider that vector **u**, the speed of the center of gravity of waste material with respect to the center of the nozzle exit section, is parallel to the longitudinal axis of the rocket and vector **b**. Consequently,

$$b \times u = 0. \tag{9.29}$$

Substituting expression (9.27) in equation (9.26), and using equation (9.28) and (9.29), we obtain

$$\frac{dL}{dt} + \dot{m} b \times (\omega \times b) = M. \tag{9.30}$$

We will designate by $\mathbf{w}_\nu$ the speed which the particle $\mathbf{m}_\nu$ would have if it were rigidly joined with the body of the rocket, and by $\mathbf{u}_\nu$, the speed of this particle with respect to the body so that

$$v_\nu = w_\nu + u_\nu. \tag{9.31}$$

On the basis of (9.19) we can write the following expression for **L** correct to the infinitesimal:

$$L = \sum_\nu m_\nu (r_\nu - r) \times v_\nu = \sum_\nu m_\nu (r_\nu - r) \times w_\nu + \\ + \sum_\nu m_\nu (r_\nu - r) \times u_\nu. \tag{9.32}$$

The first component in (9.32) constitutes the angular momentum of the rocket $\mathbf{L}_T$ on the assumption that it moves as a solid. It is known that this angular momentum can be represented in the form

$$L_T = A\omega_{x_1} x_1^0 + B\omega_{y_1} y_1^0 + C\omega_{z_1} z_1^0.$$

where A, B, C are moments of inertia of the rocket as a solid with respect to the principal axes O_1x_1, O_1y_1, O_1z_1; ω_{x_1}, ω_{y_1}, ω_{z_1} are projections of angular velocity of the body of the rocket on these axes. In virtue of symmetry of the rocket

$$B = C. \tag{9.33}$$

The second component in (9.32) is the angular momentum $\mathbf{L}_T$ of masses moving with respect to the body of the rocket in this relative motion. It would have been possible to separate from this moment the separate components, for example, the angular momentums of rapidly revolving masses inside the rocket, the angular momentums of liquid found in tanks of the rocket, and so forth. But we will disregard all the angular momentum $\mathbf{L}_T$ from those considerations that particles having great relative speed $\mathbf{u}_\nu$ consists of a very little part of the total mass of the rocket, and the majority of the particles moves at small relative speeds. It is important to consider these additional moments connected with the mobility of separate masses inside the rocket and also with deformations of the body with a detailed study of the oscillatory motion of the rocket. However, in ballistics only such oscillatory processes are important whose period is of the same order as the duration of the powered section, and in such slow processes the rocket can be completely examined as a solid.

Thus, we will consider that

$$L = L_T = A\omega_{x_1} x_1^0 + B\omega_{y_1} y_1^0 + C\omega_{z_1} z_1^0. \tag{9.34}$$

Here, in contrast to the solid, the principal moments of inertia with respect to time are inconstant, and therefore

$$\frac{dL}{dt}=\frac{dA}{dt}\omega_{x_1}x_1^0+A\frac{d\omega_{x_1}}{dt}x_1^0+A\omega_{x_1}\frac{dx_1^0}{dt}+\frac{dB}{dt}\omega_{y_1}y_1^0+$$
$$+B\frac{d\omega_{y_1}}{dt}y_1^0+B\omega_{y_1}\frac{dy_1^0}{dt}+\frac{dC}{dt}\omega_{z_1}z_1^0+C\frac{d\omega_{z_1}}{dt}z_1^0+C\omega_{z_1}\frac{dz_1^0}{dt}.$$

The principal moments of inertia, just as the mass of the rocket, decrease with burnout, and, consequently, their time derivatives are negative. Let us designate by $\dot{A}$, $\dot{B}$, and $\dot{C}$ the absolute quantities of these derivatives:

$$\left.\begin{aligned}\dot{A}&=\left|\frac{dA}{dt}\right|=-\frac{dA}{dt},\\ \dot{B}&=\left|\frac{dB}{dt}\right|=-\frac{dB}{dt},\\ \dot{C}&=\left|\frac{dC}{dt}\right|=-\frac{dC}{dt}=\dot{B}.\end{aligned}\right\} \tag{9.35}$$

As is known, derivatives of vectors x_1^0, y_1^0, z_1^0 are expressed by formulas

$$\frac{dx_1^0}{dt}=\omega\times x_1^0=(\omega_{x_1}x_1^0+\omega_{y_1}y_1^0+\omega_{z_1}z_1^0)\times x_1^0=\omega_{z_1}y_1^0-\omega_{y_1}z_1^0$$

and, analogously,

$$\frac{dy_1^0}{dt}=\omega_{x_1}z_1^0-\omega_{z_1}x_1^0,$$
$$\frac{dz_1^0}{dt}=\omega_{y_1}x_1^0-\omega_{x_1}y_1^0,$$

whence

$$\frac{dL}{dt}=\left[A\frac{d\omega_{x_1}}{dt}-(B-C)\omega_{y_1}\omega_{z_1}-\dot{A}\omega_{x_1}\right]x_1^0+$$
$$+\left[B\frac{d\omega_{y_1}}{dt}-(C-A)\omega_{z_1}\omega_{x_1}-\dot{B}\omega_{y_1}\right]y_1^0+$$
$$+\left[C\frac{d\omega_{z_1}}{dt}-(A-B)\omega_{x_1}\omega_{y_1}-\dot{C}\omega_{z_1}\right]z_1^0. \tag{9.36}$$

The second component on the left side of equation (9.30) will be transformed in the following way:

$$\dot{m}b\times(\omega\times b)=\dot{m}(-bx_1^0)\times[(\omega_{x_1}x_1^0+\omega_{y_1}y_1^0+\omega_{z_1}z_1^0)\times(-bx_1^0)]=$$
$$=\dot{m}b^2x_1^0\times(\omega_{z_1}y_1^0-\omega_{y_1}z_1^0)=\dot{m}b^2(\omega_{y_1}y_1^0+\omega_{z_1}z_1^0). \tag{9.37}$$

Finally equation (9.30) will be thus transformed:

$$\left[A\frac{d\omega_{x_1}}{dt}-(B-C)\omega_{y_1}\omega_{z_1}-\dot{A}\omega_{x_1}\right]x_1^0+$$
$$+\left[B\frac{d\omega_{y_1}}{dt}-(C-A)\omega_{z_1}\omega_{x_1}+(\dot{m}b^2-\dot{B})\omega_{y_1}\right]y_1^0+$$
$$+\left[C\frac{d\omega_{z_1}}{dt}-(A-B)\omega_{x_1}\omega_{y_1}+(\dot{m}b^2-\dot{C})\omega_{z_1}\right]z_1^0=M. \tag{9.38}$$

§ 10. Reaction Force and Moments

Let us write the equation of motion of the center of gravity of the rocket (9.18) in the form

$$m\frac{dv}{dt} = F - \dot{m}u - 2\dot{m}\dot{b} - \ddot{m}b. \tag{10.1}$$

Comparing it with the equation of motion of a solid

$$m\frac{dv}{dt} = F, \tag{10.2}$$

we see that the center of gravity of the rocket moves just as the center of gravity of a solid with a mass equal to the mass of the rocket, on which acts, besides forces having an effect on the rocket, the force

$$P_R = -(\dot{m}u + 2\dot{m}\dot{b} + \ddot{m}b). \tag{10.3}$$

This force will be called reaction force (or dynamic thrust). The equation of motion of the center of gravity of the rocket can now be written thus:

$$m\frac{dv}{dt} = F + P_R. \tag{10.4}$$

We will apply this equation to the case of operation of the rocket on the stand (without jet vanes).

On the basis of (9.11) it is possible to write: $v = -\dot{b}$, since the center of the nozzle exit section is motionless. Hence

$$\frac{dv}{dt} = -\ddot{b}. \tag{10.5}$$

From external forces of the rocket act gravity G, static thrust P_{CT}, which is equal according to the formula (5.32) to

$$P_{ct} = S_a(p_a - p)x_1^0,$$

and the reaction of supports of the stand Q, so that

$$F = G + P_{ct} + Q. \tag{10.6}$$

Substituting (10.3), (10.5) and (10.6) into equation (10.4), we will obtain

$$-m\ddot{b} = G + P_{ct} + Q - \dot{m}u - 2\dot{m}\dot{b} - \ddot{m}b,$$

whence

$$Q + G + P_{ct} - \dot{m}u + m\ddot{b} + 2\frac{dm}{dt}\dot{b} + \frac{d^2m}{dt^2}b = 0,$$

or

$$Q + G + P_{ct} - \dot{m}u + \frac{d^2(mb)}{dt^2} = 0. \tag{10.7}$$

Equation (10.7) shows that on the stand supports, besides the weight of the rocket, there acts the force

$$P = -\dot{m}u + S_a(p_a - p)x_1^0 + \frac{d^2(mb)}{dt^2}, \tag{10.8}$$

which we will call thrust.

Examining different concrete cases, it is easy to be convinced of the fact that terms $\ddot{m}\mathbf{b}$, $2\dot{m}\dot{\mathbf{b}}$, $m\ddot{\mathbf{b}}$ and $d^2(m\mathbf{b})/dt^2$ are very small as compared to other terms in formulas (10.3) and (10.8). Therefore, henceforth we will disregard them use the following expressions for reactive force and thrust:

$$P_{\text{r}} = -\dot{m}u. \tag{10.9}$$

$$P = -\dot{m}u + S_a(p_a - p)x_1^0 = P_{\text{r}} + P_{\text{ст}}. \tag{10.10}$$

Exit velocity **u** was defined above as the speed of the center of gravity of waste material with respect to the center of the nozzle exit section of the engine. But the waste material, having an infinitesimal mass, possesses final dimensions and in turn can be split into particles $dm_\varkappa$, moving at different speeds $\mathbf{u}_\varkappa$ relative to those points of the nozzle exit section through which they pass. Therefore, the concept of exit velocity **u** should be definitized.

Every particle $dm_\varkappa$ will imagine as a mass passing through the element of area of the nozzle exit section $dS_\varkappa$. Designating the density of gases in the volume occupied by the particle $dm_\varkappa$, by $\rho_\varkappa$, we can express the mass and momentum of this particle in the form

$$dm_\varkappa = \rho_\varkappa(u_\varkappa \cdot dS_\varkappa)\,dt,$$

$$u_\varkappa\,dm_\varkappa = \rho_\varkappa u_\varkappa(u_\varkappa \cdot dS_\varkappa)\,dt,$$

where $d\mathbf{S}_\varkappa$ is the vector of external normal to the element of area $dS_\varkappa$, where

$$|dS_\varkappa| = dS_\varkappa;$$

hence the momentum of all the waste material recorded earlier as $\mathbf{u}\dot{m}\,dt$ (the question is about relative motion) is equal to

$$\sum_\varkappa \rho_\varkappa u_\varkappa(u_\varkappa \cdot dS_\varkappa)\,dt = dt\int_{S_a}\rho u(u \cdot dS).$$

Thus, by **u** we mean the quantity

$$u = \frac{1}{\dot{m}}\int_{S_a}\rho u(u \cdot dS). \tag{10.11}$$

Using expression (10.11), it is possible to present the reaction force and thrust in the form

$$P_{\text{r}} = -\int_{S_a}\rho u(u \cdot dS), \tag{10.12}$$

$$P = \int_{S_a}[p\,dS - \rho u(u \cdot dS)] - S_a p x_1^0. \tag{10.13}$$

In formula (10.13) $\mathbf{S_a p_a x_1^0}$ is replaced by a more exact expression

$$\int_{S_a} p\,dS,$$

where **p** denotes the pressure of gases on the elementary area of the nozzle exit section.

Earlier it was already mentioned that the vector **u** is considered directed along the longitudinal axis of the rocket from the summit to the tail.

Consequently, the reaction force and thrust constitute vectors effective along the longitudinal axis of the rocket in a direction toward the summit. Values of these vectors are equal to

$$P_{\text{р}} = \dot{m}u, \tag{10.14}$$

$$P = \dot{m}u + S_a(p_a - p). \tag{10.15}$$

Gas-dynamic calculations and experiments show that for engines with $p_a > 0.8\ p_0$ with a change in operating conditions in not very great limits, it is possible to consider the exit velocity u constant and the pressure p_a variable directly proportional to the flow rate per second $\dot{m}$, both of these values not depending on the external pressure p. Let us call the quantity

$$u' = u + S_a \frac{p_a}{\dot{m}} = \text{const}$$

the effective exit velocity. Then

$$P = \dot{m}u' - S_a p. \tag{10.16}$$

This formula describes the dependence of thrust on the flow rate per second and on the external pressure, i.e., can be assumed as a basis of both the throttle and altitude performance of the engine. In particular, the thrust of the engine in a vacuum is equal to

$$P_{\text{п}} = \dot{m}u', \tag{10.17}$$

i.e., directly proportional to the flow rate per second, and the thrust of the engine on earth is expressed by the formula

$$P_0 = \dot{m}u' - S_a p_0, \tag{10.18}$$

whence

$$u' = \frac{P_0 + S_a p_0}{\dot{m}_0}. \tag{10.19}$$

The thrust of the engine on earth P_0 and the flow rate per second in terrestrial conditions $\dot{m}_0$ can be determined during bench tests of the engine. The thrust in flight, depending upon the flow rate and on the external pressure, is determined by formula (10.16), where instead of the effective speed of efflux u' it is possible to substitute its expression (10.19):

$$P = \frac{\dot{m}}{\dot{m}_0}(P_0 + S_a p_0) - S_a p. \tag{10.20}$$

In particular, if the flow rate in flight remains constant, then

$$P = P_0 + S_a(p_0 - p). \tag{10.21}$$

The quantity

$$P_{\text{уд}} = \frac{P}{\dot{m}g_0} = \frac{P_0 + S_a p_0}{\dot{m}_0 g_0} - \frac{S_a p}{\dot{m}g_0} = \frac{u'}{g_0} - \frac{S_a p}{\dot{m}g_0}$$

will be called specific thrust.

Specific thrust in a vacuum is equal to

$$P_{\text{уд.п}} = \frac{P_0 + S_a p_0}{\dot{m}_0 g_0} = \frac{u'}{g_0} = \text{const}$$

(does not depend on the flow rate), and at an arbitrary altitude

$$P_{y1} = P_{y1\,a} - \frac{S_a p}{\dot m g_0} = \frac{u'}{g_0} - \frac{S_a p}{\dot m g_0}.$$

In particular, on earth

$$P_{y1\,0} = \frac{u'}{g_0} - \frac{S_a p_0}{\dot m_3 g_0}.$$

Let us now compare the equation derived in § 9 of the motion of the rocket about the center of masses (9.38)

$$\left[A\frac{d\omega_{x_1}}{dt} - (B - C)\omega_{y_1}\omega_{z_1}\right]x_1^0 + \left[B\frac{d\omega_{y_1}}{dt} - (C - A)\omega_{x_1}\omega_{z_1}\right]y_1^0 +$$
$$+ \left[C\frac{d\omega_{z_1}}{dt} - (A - B)\omega_{x_1}\omega_{y_1}\right]z_1^0 =$$
$$= M + \dot A\omega_{x_1}x_1^0 - (\dot m b^2 - \dot B)\omega_{y_1}y_1^0 - (\dot m b^2 - \dot C)\omega_{z_1}z_1^0 \qquad (10.22)$$

with the Euler equation of motion of a solid about the center of gravity

$$\left[A\frac{d\omega_{x_1}}{dt} - (B - C)\omega_{y_1}\omega_{z_1}\right]x_1^0 + \left[B\frac{d\omega_{y_1}}{dt} - (C - A)\omega_{x_1}\omega_{z_1}\right]y_1^0 +$$
$$+ \left[C\frac{d\omega_{z_1}}{dt} - (A - B)\omega_{x_1}\omega_{y_1}\right]z_1^0 = M. \qquad (10.23)$$

A comparison shows that the motion of the rocket about the center of gravity occurs just as the motion of a solid with those main moments of inertia, which for the rocket at the given moment of time on which act, besides moments having an effect on the rocket, the moments

$$\left.\begin{aligned} M_{Rx_1} &= \dot A\omega_{x_1}x_1^0, \\ M_{Ry_1} &= -(\dot m b^2 - \dot B)\omega_{y_1}y_1^0, \\ M_{Rz_1} &= -(\dot m b^2 - \dot C)\omega_{z_1}z_1^0. \end{aligned}\right\} \qquad (10.24)$$

These moments will be called reaction moments.

The first of these moments is the swaying, since it acts in the direction of rotation of the rocket about the axis O_1x_1, but it is very small and, furthermore, is to a considerable degree compensated by a damping action of waste material unaccounted for by us, which removes with itself a certain angular momentum. Subsequently we will not consider it. Two other reaction moments, as is easily verified, are damping moments.

Let us transform the coefficient entering the expression for these moments to a more convenient form

$$\dot m b^2 - \dot B = \dot m b^2 - \dot C = \frac{dB}{dt} - b^2\frac{dm}{dt}.$$

The moment of inertia of the rocket with respect to the lateral axis passing through the center of gravity can be expressed in terms of the moment of inertia with respect to the parallel axis lying in the plane of the exit nozzle section:

$$B = B_a - mb^2. \qquad (10.25)$$

whence

$$\frac{dB}{dt} = \frac{dB_a}{dt} - 2mb\frac{db}{dt} - b^2\frac{dm}{dt}$$

and

$$\dot{m}b^2 - \dot{B} = \frac{dB_a}{dt} - 2b\left(m\frac{db}{dt} + b\frac{dm}{dt}\right) = \frac{dB_a}{dt} - 2b\frac{d(mb)}{dt}. \qquad (10.26)$$

Derivatives entering into the last equality can easily be calculated. We will examine separately liquids expending in flight of the rocket from every tank. Let us designate the absolute value of the flow rate per second of liquid from the i-th tank by $\dot{m}_i$ and the level of liquid above the nozzle section by h_i (we consider the mirror of the liquid parallel to the plane of the nozzle exit section). Then for the time dt from the surface of the liquid there will be expended the mass $\dot{m}_i$ dt, and inside the volume remaining filled with liquid the distribution of masses can be considered constant. Consequently, if we disregard the intrinsic moments of inertia of expended masses of liquid, the change in the moment of inertia B_a will be equal to

$$dB_a = -\sum_i h_i^2 \dot{m}_i\, dt,$$

and the change in static moment with respect to the same axis is equal to

$$d(mb) = -\sum_i h_i \dot{m}_i\, dt.$$

Substituting these expressions in equation (10.26), we will obtain

$$\dot{m}b^2 - \dot{B} = -\sum_i h_i^2 \dot{m}_i + 2b\sum_i h_i \dot{m}_i = \sum_i h_i(2b - h_i)\dot{m}_i. \qquad (10.27)$$

Hence we obtain convenient expressions for reaction damping moments with respect to lateral axes of the rocket:

$$\left.\begin{aligned} M_{Ry_1} &= -\omega_{y_1}\sum_i h_i(2b - h_i)\dot{m}_i, \\ M_{Rz_1} &= -\omega_{z_1}\sum_i h_i(2b - h_i)\dot{m}_i. \end{aligned}\right\} \qquad (10.28)$$

The $\sum_i h_i(2b - h_i)\dot{m}_i$ will for brevity be designated m_R^{ω}.

Let us explain the physical essence of the reaction force and reaction moments.

Gases passing from the nozzle of the engine have a speed u with respect to the body and $u + \dot{b}$ with respect to the center of gravity of the rocket. Value $\dot{m}(u + b)$ constitutes a force which must be applied to these gases in order to impart to them such speed. In virtue of the third law of Newton, to the center of gravity of the rocket on the side of the gases will be applied the force $\dot{m}(u + \dot{b})$, which is the main term of the reaction force.

In order to understand the origin of the other two terms in the formula for reaction force, it is necessary to comprehend the origin of the quantity $\dot{m}b$ in expression (9.9) for momentum of the rocket. This quantity is nothing else but the momentum of expended mass with respect to the center of gravity of the rocket. Actually, in a closed system whose mass is not expended, the momentum coincides with the momentum of the material particle with a mass equal to the mass of the system and with the motion identical to the motion of the center of gravity of the system. It is a different matter in the case of the rocket — system with a variable more accurate with the expending mass.

If at the moment of time t we consider all its mass concentrated at one point, then after the time dt all this mass m will shift at a distance v dt, but, furthermore, as a result of the internal motion part of the mass $\dot{m}$ dt will shift at a distance b from the new position of the center of gravity. The first shift corresponds to the momentum

$$\frac{mv\,dt}{dt} = mv,$$

and the second to momentum

$$\frac{\dot m\, db}{dt} = \dot m \dot b.$$

It is clear that for an increase in the second component of momentum there is necessary a force equal to

$$\frac{d(\dot m b)}{dt} = \dot m \dot b + \bar m b.$$

Since this force acts on the increase in momentum of masses in the rocket with respect to the center of gravity, then according to the third law of Newton the opposite force, $\dot m \dot b$ - $\bar m b$ should be applied from the side of moving masses to the center of gravity of the rocket.

In the same way it is easy to explain the origin of reaction moments. Let us consider, for example, the rotation about the lateral axis O_1y_1.

Owing to the efflux of gases the angular momentum of the rocket decreases by the magnitude

$$|dB|\omega_{y_1} = \dot B \omega_{y_1} dt.$$

But the very gases passing from the rocket possess, with respect to the center of gravity of the rocket, the angular momentum

$$\dot m b^2 \omega_{y_1} dt.$$

Consequently, during the time dt they obtain, because of the rocket, the angular momentum

$$\dot m b^2 \omega_{y_1} dt - \dot B \omega_{y_1} dt,$$

for which to them there should be applied from the side of the rocket the moment of forces

$$(\dot m b^2 - \dot B)\omega_{y_1}.$$

In virtue of the law of counteraction on the rocket from the side of the exhaust gases, there should act the moment about axis O_1y_1

$$-(\dot m b^2 - \dot B)\omega_{y_1},$$

which is the reaction moment.

In summarizing, we can say that the reaction force (moment) is equal in value and opposite in direction to the force (moment) which must be applied to gases passing from the rocket for a change in their momentum (angular momentum).

Let us sum up our whole analysis of forces and moments having an effect on the rocket and the derivation of equations of motion.

We established that for the rocket it is possible to use equations of motion having the form of equations of motion of a solid (10.2) and (10.23) if to the forces having an effect on the rocket from without — gravity, aerodynamic forces, static thrust and forces from controls, — we join the reaction force, and to external moments — aerodynamic destabilizing or restoring moment — we join reaction moments. The reaction force together with the static thrust we united (in the first approximation) into a single tractive force.

Now it remained to pass from the vector form of equations of motion to the coordinate form, for which it will be required to find component forces and moments having an effect on the rocket along axes of coordinates.

§ 11. Resolution of Forces and Moments Along Axes of Coordinates

To determine the vectorial components along axes of coordinates it is necessary to know the direction cosines of this vector. The direction cosines of unit vectors of axes of the bound system of coordinates, with respect to the terrestrial system of coordinates, are already well-known. Thereby we determine in both systems of coordinates by means of formulas (2.6) and (2.7) direction cosines of all forces and moments effective along axes of the bound system of coordinates, namely, tractive forces comprised of the reaction force and static thrust, forces and moments from controls, aerodynamic damping moments and reaction moments.

Gravity acts in the direction opposite to the direction of the radius vector of the rocket

$$r = xx^0 + (R+y)y^0 + zz^0$$

and, consequently, has along axes of the terrestrial system of coordinates the following direction cosines (Table 11.1):

Table 11.1

	Ox	Oy	Oz
g	$-\frac{x}{r}$	$-\frac{R+y}{r}$	$-\frac{z}{r}$

It remains to find direction cosines of aerodynamic forces and their moments, which both in magnitude and in terms of direction depend on the velocity vector of the rocket v relative to the terrestrial system of coordinates.

The direction of the velocity vector, in other words, the direction of the tangent to the trajectory, will be determined by two angles θ and σ, which are determined in the following way. Let us draw through the velocity vector an inclined plane perpendicular to plane Oxy (Fig. 11.1). The angle composed by this plane with plane Oxz will be designated by θ, and the angle between the velocity vector and plane Oxy by σ. These angles are analogous to angles φ and ξ determining the direction of the longitudinal axis of the rocket relative to the terrestrial system of the coordinates. We will consider the angle θ positive if the velocity vector is directed upwards, and the angle σ if this vector is directed to the left of the plane Oxy. Then the vectorial component of speed along the axis Oz will be equal to

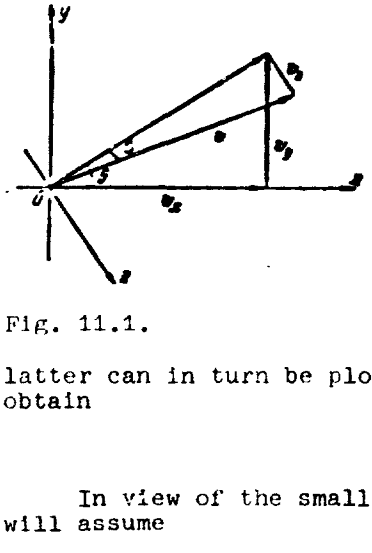

Fig. 11.1.

$$v_z = -v \sin \sigma, \tag{11.1}$$

and the projection on plane Oxy to -v cos σ. The latter can in turn be plotted on axes Ox and Oy, as a result of which we will obtain

$$v_x = v \cos \sigma \cos \theta, \quad v_y = v \cos \sigma \sin \theta. \tag{11.2}$$

In view of the smallness of angle σ during normal flight of the rocket, we will assume

$$\cos \sigma = 1, \quad \sin \sigma = \sigma, \tag{11.3}$$

whence

$$v_x = v \cos \theta, \quad v_y = v \sin \theta, \quad v_z = -v\sigma. \tag{11.4}$$

Hence we find the direction cosines of vector v and opposite to it the vector of X (Table 11.2).

Table 11.2

	Ox	Oy	Oz
v	$\cos\theta$	$\sin\theta$	$-\alpha$
X	$-\cos\theta$	$-\sin\theta$	α

It was already mentioned (§ 7) that the vector of the aerodynamic moment M_a is perpendicular both to the longitudinal axis of the rocket and also to the tangent to the trajectory and, consequently, differs only by a numerical factor from the vector product of unit vectors x_1^0 and v^0, where if this is the restoring moment then it is directed along vector $x_1^0 \times v^0$, and if it is the destabilizing moment in the opposite direction. Noticing that the modulus of the vector product $x_1^0 \times v^0$, by definition is equal to $\sin\alpha \approx \alpha$, and the quantity of the aerodynamic moment on the basis of formula (7.1) is equal to

$$M_a = c'_{y_1} qS(x_d - x_T)\alpha$$

(for the restoring moment $M_a > 0$, for destabilizing, $M_a < 0$), we obtain that the vector of aerodynamic moment can be thus expressed:

$$M_a = c'_{y_1} qS(x_d - x_T)\alpha \frac{x_1^0 \times v^0}{\alpha} = c'_{y_1} qS(x_d - x_T)(x_1^0 \times v^0). \tag{11.5}$$

Lift Y is perpendicular to vectors v^0 and $x_1^0 \times v^0$ and coincides in direction with vector $v^0 \times (x_1^0 \times v^0)$. The modulus of this vector is also equal to α (since vectors v^0 and $x_1^0 \times v^0$ are mutually perpendicular), and therefore lift can be represented in the following way:

$$Y = c'_{y_1} qS\alpha \frac{v^0 \times (x_1^0 \times v^0)}{\alpha} = c'_{y_1} qSv^0 \times (x_1^0 \times v^0). \tag{11.6}$$

We use equality $x_1^0 = y_1^0 \times z_1^0$ and transform the vector product $x_1^0 \times v^0$ to the form

$$x_1^0 \times v^0 = (y_1^0 \times z_1^0) \times v^0 = (v^0 \cdot y_1^0) z_1^0 - (v^0 \cdot z_1^0) y_1^0.$$

Then expressions (11.5) and (11.6) will take the form

$$M_a = c'_{y_1} qS(x_d - x_T)[(v^0 \cdot y_1^0) z_1^0 - (v^0 \cdot z_1^0) y_1^0] = M_{az_1} + M_{ay_1}. \tag{11.7}$$

$$Y = c'_{y_1} qSv^0 \times [(v^0 \cdot y_1^0) z_1^0 - (v^0 \cdot z_1^0) y_1^0] = Y_y + Y_z, \tag{11.8}$$

where

$$\left.\begin{aligned} M_{az_1} &= c'_{y_1} qS(x_d - x_T)(v^0 \cdot y_1^0) z_1^0, \\ M_{ay_1} &= -c'_{y_1} qS(x_d - x_T)(v^0 \cdot z_1^0) y_1^0. \end{aligned}\right\} \tag{11.9}$$

$$\left.\begin{aligned}Y_y&=c'_{y_1}qS(v^0\cdot y_1^0)(v^0\times z_1^0),\\ Y_z&=-c'_{y_1}qS(v^0\cdot z_1^0)(v^0\times y_1^0).\end{aligned}\right\}\tag{11.10}$$

Decompositions of (11.7) and (11.8) have geometric meaning: lift is presented in the form of the sum of two forces directed along the normal to the trajectory, one of which, Y_y, lies in plane $O_1x_1y_1$, the other, Y_z, in plane $O_1x_1z_1$, and the aerodynamic moment is replaced by the sum of moments of these forces. Convenience of such decomposition consists in the fact that for vectors Y_y and Y_z, and expecially for M_{ay_1} and M_{az_1}, it is easy to find their magnitudes and direction cosines. Actually, vectors M_{ay_1} and M_{az_1} are already represented in the form of products of scalars

$$M_{ay_1}=-c'_{y_1}qS(x_2-x_T)(v^0\cdot z_1^0)\tag{11.11}$$

and

$$M_{az_1}=c'_{y_1}qS(x_2-x_T)(v^0\cdot y_1^0)\tag{11.12}$$

on unit vectors y_1^0 and z_1^0.

It is found that vectors $(v^0 \times z_1^0)$ and $(v^0 \times y_1^0)$ entering into expressions for Y_y and Y_z with an accuracy accepted by us can also be considered unity. Really, if we disregard the square and pairwise products of small angles ξ, η and σ, and for the small angle $\varphi - \theta$ we assume

$$\left.\begin{aligned}\sin(\varphi-\theta)&=\varphi-\theta,\\ \cos(\varphi-\theta)&=1,\end{aligned}\right\}\tag{11.13}$$

then

$$\begin{aligned}v^0\times z_1^0&=\begin{vmatrix}x^0 & y^0 & z^0\\ \cos\theta & \sin\theta & -\sigma\\ \xi\cos\varphi+\eta\sin\varphi & \xi\sin\varphi-\eta\cos\varphi & 1\end{vmatrix}=\\ &=(\sin\theta+\xi\sigma\sin\varphi-\eta\sigma\cos\varphi)x^0+\\ &\quad+(-\cos\theta-\xi\sigma\cos\varphi-\eta\sigma\sin\varphi)y^0+\\ &\quad+[(\xi\sin\varphi-\eta\cos\varphi)\cos\theta-(\xi\cos\varphi+\eta\sin\varphi)\sin\theta]z^0\approx\\ &\approx x^0\sin\theta-y^0\cos\theta-z^0\eta.\end{aligned}\tag{11.14}$$

$$\begin{aligned}v^0\times y_1^0&=\begin{vmatrix}x^0 & y^0 & z^0\\ \cos\theta & \sin\theta & -\sigma\\ -\sin\varphi & \cos\varphi & \eta\end{vmatrix}=\\ &=(\eta\sin\theta+\sigma\cos\varphi)x^0+(-\eta\cos\theta+\sigma\sin\varphi)y^0+\\ &\quad+\cos(\varphi-\theta)z^0\approx x^0(\eta\sin\theta+\sigma\cos\varphi)+\\ &\quad+y^0(-\eta\cos\theta+\sigma\sin\varphi)+z^0.\end{aligned}\tag{11.15}$$

For quantities of these vectors the following equalities are correct

$$\begin{aligned}|v^0\times z_1^0|&=\sqrt{\sin^2\theta+\cos^2\theta+\eta^2}\approx 1.\\ |v^0\times y_1^0|&=\\ &=\sqrt{(\eta\sin\theta+\sigma\cos\varphi)^2+(-\eta\cos\theta+\sigma\sin\varphi)^2+1}\approx 1.\end{aligned}$$

We write expressions (11.10) in the form

$$Y_y = -c'_{y_1} qS(v^0 \cdot y_1^0)(z_1^0 \times v^0),$$
$$Y_z = c'_{y_n} qS(v^0 \cdot z_1^0)(y_1^0 \times v^0),$$

and scalar factors with vectorx $z_1^0 \times v^0$ and $y_1^0 \times v^0$ will be taken for quantities of forces Y_y and Y_z:

$$\left.\begin{aligned} Y_y &= -c'_{y_n} qS(v^0 \cdot y_1^0), \\ Y_z &= c'_{y_n} qS(v^0 \cdot z_1^0). \end{aligned}\right| \quad (11.16)$$

Expressions (11.11), (11.12) and (11.16) show that scalar products

$$v^0 \cdot y_1^0 = \cos\theta(-\sin\varphi) + \sin\theta\cos\varphi - \alpha\eta \approx \sin(\theta - \varphi) \approx \theta - \varphi$$

and

$$v^0 \cdot z_1^0 = \cos\theta(\xi\cos\varphi + \eta\sin\varphi) + \sin\theta(\xi\sin\varphi - \eta\cos\varphi) - \sigma \approx$$
$$\approx \xi\cos(\varphi - \theta) + \eta\sin(\varphi - \theta) - \sigma \approx \xi - \sigma$$

can be examined as numerical values of angles of attack in planes $O_1x_1y_1$ and $O_1x_1z_1$, which we designate by α_y and α_z (with a change in the sign for $v^0 \cdot y_1^0$):

$$\alpha_y = -v^0 \cdot y_1^0 = \varphi - \theta, \quad (11.17)$$
$$\alpha_z = v^0 \cdot z_1^0 = \xi - \sigma. \quad (11.18)$$

With these designations formulas (11.11), (11.12) and (11.16) will take the following form:

$$\left.\begin{aligned} M_{ay_1} &= -c'_{y_n} qS(x_d - x_t)\alpha_z, \\ M_{az_1} &= -c'_{y_n} qS(x_d - x_t)\alpha_y; \end{aligned}\right| \quad (11.19)$$

$$\left.\begin{aligned} Y_y &= c'_{y_n} qS\alpha_y, \\ Y_z &= c'_{y_n} qS\alpha_z. \end{aligned}\right| \quad (11.20)$$

These formulas determine magnitudes of moments M_{ay_1}, M_{az_1}, and forces Y_y and Y_z. Direction cosines of these vectors coincide with components of unit vectors y_1^0, z_1^0, $z_1^0 \times v^0$ and $y_1^0 \times v^0$ respectively.

§ 12. Summary of Formulas for Forces and Moments Having an Effect on the Rocket

For the best clarity of formulas derived above let us put them into Tables 12.1 and 12.2.

Table 12.1

Force	Quantity	Cosine of angle with angle		
		Ox	Oy	Oz
Gravity	$G = mg$	$-\frac{x}{r}$	$-\frac{R+y}{r}$	$-\frac{z}{r}$
Drag	$X = c_x qS$	$-\cos\theta$	$-\sin\theta$	0
Lift	$Y_y = c'_{y_1} qS\alpha_y$	$-\sin\theta$	$\cos\theta$	η
Lift	$Y_z = c'_{y_1} qS\alpha_z$	$-\eta\sin\theta - \sigma\cos\varphi$	$\eta\cos\theta - \sigma\sin\varphi$	-1
Axial force from jet vanes	$X_{1p} = 4Q_{p0} + \lambda(\delta_1^2 + 2\delta_2^2 + \delta_3^2)$	$-\cos\varphi$	$-\sin\varphi$	ξ
Lateral force from jet vanes 2 and 4	$Y_{1p} = 2R'\delta_2$	$-\sin\varphi$	$\cos\varphi$	η
Lateral force from jet vanes 1 and 3	$Z_{1p} = R'(\delta_1 + \delta_3)$	$\xi\cos\varphi + \eta\sin\varphi$	$\xi\sin\varphi - \eta\cos\varphi$	1
Thrust	$P = \frac{\dot{m}}{m_0}(P_0 + S_a p_0) - S_a p$	$\cos\varphi$	$\sin\varphi$	$-\xi$

Table 12.2

Moment	Quantity	Cos of angle with angle		
		O_1x_1	O_1y_1	O_1z_1
Aerodynamic moment	$M_{Ay_1} = -c'_{y_1} qS(x_2 - x_1)\alpha_z$	0	1	0
Aerodynamic moment	$M_{Az_1} = -c'_{y_1} qS(x_2 - x_1)\alpha_y$	0	0	1
Moment of jet vanes 1 and 3 with respect to axis O_1x_1	$M_{x_1} = R'h_1(\delta_3 - \delta_1)$	1	0	0
Moment of jet vanes 1 and 3 with respect to axis O_1y_1	$M_{y_1} = R'(l_1 - x_1)(\delta_1 + \delta_3)$	0	1	0
Moment of jet vanes 2 and 4 with respect to axis O_1z_1	$M_{z_1} = -2R'(l_1 - x_1)\delta_2$	0	0	1
Damping moment	$\Delta M_{x_1} = -m_{x_1}^{\omega} Sl^2 \rho v\omega_{x_1}$	1	0	0
Damping moment	$\Delta M_{y_1} = -m_{y_1}^{\omega} Sl^2 \rho v\omega_{y_1}$	0	1	0
Damping moment	$\Delta M_{z_1} = -m_{z_1}^{\omega} Sl^2 \rho v\omega_{z_1}$	0	0	1
Reaction moment	$M_{Ry_1} = -m_R^{\omega}\omega_{y_1}$	0	1	0
Reaction moment	$M_{Rz_1} = -m_R^{\omega}\omega_{z_1}$	0	0	1

§ 13. Equations of Motion in Coordinate Form

Let us project equation (10.2) on the axis of the terrestrial system of coordinates. With this by dv/dt one should understand the full acceleration of the center of gravity of the rocket in absolute motion determined by formula (1.8), and in vector F unite all forces reduced in Table 12.1. As a result we obtain:

$$
\begin{aligned}
m(\ddot{x} + J_{ex} + J_{cx}) = {} & -mg\frac{x}{r} - c_x qS\cos\theta - \\
& - c'_{y_1}qS\left[\alpha_y \sin\theta + \alpha_z(\eta\sin\theta + \sigma\cos\varphi)\right] - X_{1p}\cos\varphi - \\
& - 2R'\delta_2\sin\varphi + R'(\delta_1 + \delta_3)(\xi\cos\varphi + \eta\sin\varphi) + P\cos\varphi. \\
m(\ddot{y} + J_{ey} + J_{cy}) = {} & -mg\frac{R+y}{r} - c_x qS\sin\theta + \\
& + c'_{y_1}qS\left[\alpha_y\cos\theta + \alpha_z(\eta\cos\theta - \sigma\sin\varphi)\right] - X_{1p}\sin\varphi + \\
& + 2R'\delta_2\cos\varphi + R'(\delta_1 + \delta_3)(\xi\sin\varphi - \eta\cos\varphi) + P\sin\varphi. \\
m(\ddot{z} + J_{ez} + J_{cz}) = {} & -mg\frac{z}{r} + c_x qS\sigma + c'_{y_1}qS(\alpha_y\eta - \alpha_z) + \\
& + X_{1p}\xi + 2R'\delta_2\eta + R'(\delta_1 + \delta_3) - P\xi.
\end{aligned}
$$

In these equations we can disregard terms containing pairwise products of small angles, including the products of angles ξ and η on angles of deflection of the control surfaces, as a result of which equation will take following form:

$$
\left.
\begin{aligned}
\ddot{x} = {} & \frac{1}{m}\left[(P - X_{1p})\cos\varphi - c_x qS\cos\theta - c'_{y_1}qS(\varphi - \theta)\sin\theta - \right. \\
& \left. - 2R'\delta_2\sin\varphi\right] - \frac{x}{r}g - J_{ex} - J_{cx}. \\
\ddot{y} = {} & \frac{1}{m}\left[(P - X_{1p})\sin\varphi - c_x qS\sin\theta + c'_{y_1}qS(\varphi - \theta)\cos\theta + \right. \\
& \left. + 2R'\delta_2\cos\varphi\right] - \frac{R+y}{r}g - J_{ey} - J_{cy}. \\
\ddot{z} = {} & -\frac{1}{m}\left[(P - X_{1p})\xi - c_x qS\sigma + c'_{y_1}qS(\xi - \sigma) - \right. \\
& \left. - R'(\delta_1 + \delta_3)\right] - \frac{z}{r}g - J_{ez} - J_{cz}.
\end{aligned}
\right| \qquad (13.1)
$$

The vector equation (10.23) is more convenient to project on the axis of the bound system of coordinates. In this equation it is necessary to include in M all moments given in Table 12.2. Projecting and considering that B = C, we will obtain:

$$
\left.
\begin{aligned}
& A\frac{d\omega_{x_1}}{dt} = R'h_1(\delta_3 - \delta_1) - m_{x_1}^{\omega}Sl^2\rho v\omega_{x_1}. \\
& B\frac{d\omega_{y_1}}{dt} - (B - A)\omega_{x_1}\omega_{z_1} = -c'_{y_1}qS(x_\text{д} - x_\text{т})(\xi - \sigma) + \\
& \qquad + R'(l_1 - x_\text{т})(\delta_1 + \delta_3) - m_{y_1}^{\omega}Sl^2\rho v\omega_{y_1} - m_R^{\omega}\omega_{y_1}. \\
& B\frac{d\omega_{z_1}}{dt} + (B - A)\omega_{x_1}\omega_{y_1} = -c'_{y_1}qS(x_\text{д} - x_\text{т})(\varphi - \theta) - \\
& \qquad - 2R'(l_1 - x_\text{т})\delta_2 - m_{z_1}^{\omega}Sl^2\rho v\omega_{z_1} - m_R^{\omega}\omega_{z_1}.
\end{aligned}
\right| \qquad (13.2)
$$

To equations (13.1) and (13.2) it is necessary to add the nonholonomic constraint between v, θ, σ and $\dot{x}$, $\dot{y}$, $\dot{z}$, and also between ω_{x_1}, ω_{y_1}, ω_{z_1} and $\dot{\varphi}$, ξ, $\dot{\eta}$:

$$\left.\begin{aligned} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \\ \dot{z} &= -v\sigma; \end{aligned}\right\} \tag{13.3}$$

$$\left.\begin{aligned} \omega_{x_1} &= -\dot{\varphi}\xi + \dot{\eta}, \\ \omega_{y_1} &= \dot{\varphi}\eta + \dot{\xi}, \\ \omega_{z_1} &= \dot{\varphi} - \dot{\xi}\eta. \end{aligned}\right\} \tag{13.4}$$

Tne relation (13.3) coincide with equations (11.4) and relations (13.4) with equations (2.9).

Finally, for the determination of angles of deflection of the control surfaces there are necessary these equations of control

$$\left.\begin{aligned} F_1[\delta_1, x, y, z, \varphi, \xi, \eta] &= 0, \\ F_2[\delta_2, x, y, z, \varphi, \xi, \eta] &= 0, \\ F_3[\delta_3, x, y, z, \varphi, \xi, \eta] &= 0. \end{aligned}\right\} \tag{13.5}$$

The fifteen equations (13.1)-(13.5) permit determining these functions: x, y, z, φ, ξ, η, v, θ, σ, ω_{x_1}, ω_{y_1}, ω_{z_1}, δ_1, δ_2, δ_3.

CHAPTER IV

SIMPLIFICATION OF EQUATIONS OF MOTION

§ 14. Equations of Motion for Powered-Flight Trajectory

Equations of motion obtained in § 13 can be assumed as a basis of the solution of the many problems of the dynamics of flight. But practically into these equations always are introduced those or other simplifications the essence of which is closely connected with the content of the problem to be solved.

Let us start with the derivation of equations for the solution of a definite class of problems, problems of ballistics of long range rockets. In these problems the most important magnitude subject to determination and investigation is the complete flying range. Flying range depends mainly on the trajectory of the center of gravity of the rocket. Motion of rocket about the center of gravity is examined in ballistics so far as it affects the trajectory of the center of gravity. In particular, in the solution of problems of ballistics it is possible to be distracted from the influence on trajectory of small oscillations of the rocket about the center of gravity. Thus the most important equations for us will be equations (13.1) and (13.3) and less important, equations (13.2) and (13.4) in which we will produce main simplifications.

In equations (13.1) the members depending on angles of deflection of control surfaces are secondary in their value. Therefore, equations of control (13.5) can be used in considerably simplified form. Further, in equations of motion of the center of gravity (13.1) and (13.3) the first two equations depend little on what will be the solution of the third equation (for z and $\dot{z}$), and, consequently, in certain cases equations for x and y can be examined independently of equations for z. Finally, there are possible simplifications of equations of motion as a result of rejecting certain emebers immaterial with a certain accuracy of calculations. In such a plan and we will start the simplification of equations of motion for the investigation of motion of the rocket on a powered-flight trajectory.

If we disregard small oscillations of the rocket about the center of gravity, then the motion of the rocket will be accomplished with insignificant angular speeds and accelerations. For example, the angular velocity of the slope of axis of the rocket is 0.01-0.03 1/s and is changed very slowly with the exception of separate points. Quite insignificant are angular velocities with respect to axes O_1x_1 and O_1y_1. Consequently, in equations (13.2) it is possible to disregard members proportional to angular velocities and accelerations, and these equations take the form of conditions of equilibrium between aerodynamic moments and moments from controls:

$$\left.\begin{aligned} R'h_1(\delta_3-\delta_1)&=0,\\ c'_{y_1}qS(x_д-x_т)(\psi-\sigma)-R'(l_1-x_т)(\delta_1+\delta_3)&=0,\\ c'_{y_1}qS(x_д-x_т)(\varphi-\theta)+2R'(l_1-x_т)\delta_2&=0.\end{aligned}\right\}\quad(14.1)$$

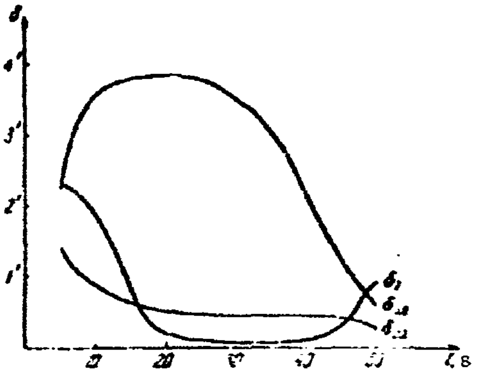

Fig. 14.1.

Figure 14.1 shows for one of the rockets values of angles of deviation of jet vanes necessary for compensation of the aerodynamic damping moment ($\delta_{вд}$), reaction moment ($\delta_{рд}$) and the surmounting of the moment of inertia (δ_1). As can be seen from the figure, all these angles are minute (range of turn of jet vanes exceeds ±20°).

From the same considerations it is possible to record the equations of control (13.5) as conditions of equilibrium between commands produced by sensing devices of the conrol system and deflections of the effectors. In the first approximation the deflections of the effectors can be considered directly proportional to the commands proceeding from the sensing elements. It is clear that zero commands correspond to zero deflections of the effectors.

As was shown in § 6, with a deflection of the axis of the rocket upwards from the program position on the potentiometer of the gyro horizon the displacement $\Delta\varphi'$ appears. We will consider that it corresponds to the proportional deflection of jet vanes 2 and 4 downwards (which we consider a positive):

$$\delta_2 = \delta_4 = a_0 \Delta\varphi'. \tag{14.2}$$

Deviation of the axis of the rocket to the left of the assigned plane of firing causes displacement ξ' of the potentiometer on the axis of the external frame of the vertical gyro. We assume that proportional to this displacement jet vanes 1 and 3 are deflected to the right (this deflection is considered negative). Finally, with the turn of the rocket about the longitudinal axis clockwise (depending on the flight) on the intermediate axis of the vertical gyro there appears a displacement of the potentiometer η' and, according to our assumption, this will induce deflection of jet vane 1 to the left and jet vane 3, to the right. Considering parameters of the control system with respect to lateral axes identical, we can write:

$$\left.\begin{aligned} \delta_1 &= -a_0\xi' + b_0\eta', \\ \delta_3 &= -a_0\xi' - b_0\eta'. \end{aligned}\right\} \tag{14.3}$$

Coefficients a_0 and b_0 characterize the sensitivity of the control system: the higher their numerical value, the greater the reaction of control surfaces on deviations of the axis of the rocket. They are called by static amplification coefficients, since they characterize the reaction of the control surfaces on a constant (or slowly changing) signal from the gyro-instruments.

Using expression (6.10)-(6.12) for ξ', η' and $\Delta\varphi'$, we will record the equation of control of the rocket in the form

$$\left.\begin{aligned} \delta_1 &= -a_0\xi + b_0\eta, \\ \delta_2 &= \delta_4 = a_0\Delta\varphi, \\ \delta_3 &= -a_0\xi - b_0\eta. \end{aligned}\right\} \tag{14.4}$$

We can now exclude angles of deflection of control surfaces from equations (14.1):

$$\left.\begin{aligned} 2R'h_1 b_0\eta &= 0, \\ c'_{y_1} qS(x_2 - x_T)(\xi - \sigma) + 2R'(l_1 - x_T) a_0\xi &= 0, \\ c'_{y_1} qS(x_2 - x_T)(\varphi - \theta) + 2R'(l_1 - x_T) a_0\Delta\varphi &= 0. \end{aligned}\right\} \tag{14.5}$$

The first of these equations gives

$$\eta = 0. \tag{14.6}$$

The second equation (14.5) can be transformed in the following way:

$$\xi = \frac{c'_{y_1} q S (x_2 - x_T)}{c'_{y_1} q S (x_2 - x_T) + 2a_0 R' (l_1 - x_T)} \sigma. \tag{14.7}$$

Finally, we transform the third equation (14.5):

$$c'_{y_1} q S (x_2 - x_T)(\varphi - \theta) + 2a_0 R' (l_1 - x_T)[(\varphi - \theta) - (\varphi_{np} - \theta)] = 0,$$

or

$$\varphi - \theta = \frac{2a_0 R' (l_1 - x_T)}{c'_{y_1} q S (x_2 - x_T) + 2a_0 R' (l_1 - x_T)} (\varphi_{np} - \theta). \tag{14.8}$$

If we designate by A the quantity

$$A = \frac{2a_0 R' (l_1 - x_T)}{c'_{y_1} q S (x_2 - x_T) + 2a_0 R' (l_1 - x_T)}, \tag{14.9}$$

then equations (14.5) lead to equations

$$\varphi - \theta = A (\varphi_{np} - \theta), \tag{14.10}$$

$$\xi = (1 - A) \sigma, \tag{14.11}$$

$$\eta = 0. \tag{14.12}$$

From equation (14.10) it follows:

$$\Delta\varphi = \varphi - \varphi_{np} = -(1 - A)(\varphi_{np} - \theta). \tag{14.13}$$

Using these equations and expressions (14.4) for angles of deflections of control surfaces, we will transform equations of motion of the center of gravity (13.1):

$$\left.\begin{aligned}
\ddot{x} &= \frac{1}{m} \Big[(P - X_{1p}) \cos\varphi - c_x q S \cos\theta - \\
&\quad - c'_{y_1} q S (\varphi - \theta) \sin\theta + 2a_0 R' (1 - A)(\varphi_{np} - \theta) \sin\varphi\Big] - \\
&\qquad - \frac{x}{r} g - j_{ex} - j_{cx}, \\
\ddot{y} &= \frac{1}{m} \Big[(P - X_{1p}) \sin\varphi - c_x q S \sin\theta + \\
&\quad + c'_{y_1} q S (\varphi - \theta) \cos\theta - 2a_0 R' (1 - A)(\varphi_{np} - \theta) \cos\varphi\Big] - \\
&\qquad - \frac{R + y}{r} g - j_{ey} - j_{cy}, \\
\ddot{z} &= - \frac{1}{m} \Big[(P - X_{1p})(1 - A) \sigma - c_x q S \sigma - \\
&\quad - c'_{y_1} q S A \sigma + 2a_0 R' (1 - A) \sigma\Big] - \frac{z}{r} g - j_{ez} - j_{cz}.
\end{aligned}\right| \tag{14.14}$$

In these equations it is expediently to turn to variables v, θ, and σ, using relation (13.3). Differentiating the latter, we will obtain

$$\left.\begin{aligned}\ddot{x}&=\dot{v}\cos\theta-v\dot{\theta}\sin\theta,\\ \ddot{y}&=\dot{v}\sin\theta+v\dot{\theta}\cos\theta,\end{aligned}\right\}\tag{14.15}$$

or, solving with respect to $\dot{v}$ and $v\dot{\theta}$,

$$\left.\begin{aligned}\dot{v}&=\ddot{x}\cos\theta+\ddot{y}\sin\theta,\\ v\dot{\theta}&=-\ddot{x}\sin\theta+\ddot{y}\cos\theta.\end{aligned}\right\}\tag{14.16}$$

Let us insert in (14.16) $\ddot{x}$ and $\ddot{y}$ from equations (14.14):

$$\begin{aligned}\dot{v}={}&\frac{1}{m}[(P-X_{1p})\cos(\varphi-\theta)-c_xqS+\\&+2a_0R'(1-A)(\varphi_{np}-\theta)\sin(\varphi-\theta)]-\\&-\left(\frac{x}{r}g+J_{ex}+J_{cx}\right)\cos\theta-\left(\frac{R+y}{r}g+J_{ey}+J_{cy}\right)\sin\theta,\end{aligned}\tag{14.17}$$

$$\begin{aligned}v\dot{\theta}={}&\frac{1}{m}[(P-X_{1p})\sin(\varphi-\theta)+c_{y_1}'qS(\varphi-\theta)-\\&-2a_0R'(1-A)(\varphi_{np}-\theta)\cos(\varphi-\theta)]+\\&+\left(\frac{x}{r}g+J_{ex}+J_{cx}\right)\sin\theta-\left(\frac{R+y}{r}g+J_{ey}+J_{cy}\right)\cos\theta.\end{aligned}\tag{14.18}$$

Up till now approximate equalities (11.13) were used only in secondary members. In order to use them in members having the largest magnitude in equations of motion, for example, in the first member of equation (14.17), it is necessary to give one self a report in the magnitude of the error committed. From tables of trigonometric functions it is easy to verify that, considering cos α_y = 1, we commit an error not exceeding

0,1% when $\alpha_y \leqslant 2^\circ 5$,
0,2% when $\alpha_y \leqslant 3^\circ 5$,
0,5% when $\alpha_y \leqslant 5^\circ$,
1% when $\alpha_y \leqslant 8^\circ$,
2% when $\alpha_y \leqslant 11^\circ$,
5% when $\alpha_y \leqslant 18^\circ$.

Since for ballistic missiles the angle of attack usually does not exceed 2-3°, and the thrust and drag of control surfaces are known correct to 1-2% of the thrust value, then the accuracy of the member $(P - X_{1p}) \cos \alpha_y$ almost will not suffer from replacement of cos α_y by unity. An even smaller error is given by replacement of sin α_y by α_y. The third member in brackets in equation (14.17) has the order α_y^2 and on the same basis can be rejected (let us note that this rejecting partially compensates replacement of cos α_y by unity in the first member).

Subsequently we will use the following principle of simplification of motion equations. If in the equation there are contained such members whose absolute value is less than the possible error in the main (in value) members, then these members can be rejected without damage to accuracy of the equation. The influence of accuracy of the equation on the accuracy of its solution is not investigated. Let us consider from this point of view members considering the attraction and rotation of the earth in equations (14.17) and (14.18). The accuracy of these equations is determined by member $(P - X_{1p})/m$, which in the beginning of flight of the rocket has a magnitude not smaller than the acceleration due to gravity $g_0 \approx 9.8$ m/s^2, and toward the end of the powered section because of the decrease in mass of the rocket it increases a few times. The accuracy of this member, as was already mentioned, is equal to 1-2%, i.e., not higher than 0.1 m/s^2. In equations (14.17) and (14.18) we will not consider members smaller than 0.05 m/s^2 or 0.005 g.

Thus, the member $\frac{x}{r}$ g can be disregarded when $x < 0.005$ r, i.e., all the more when $x \leq 0.005 R \approx 30$ km, and taking into account factors sin θ and cos θ when $x < 50$ km. The factor $(R + y)/r$ when g can be considered equal to unity when $R + y > 0.995$ r,

$$x^2 = r^2 - (R + y)^2 < 0.01r^2,$$
$$x < 0.1r \approx 600 \text{ km}.$$

Members j_{ex} and j_{ey} consitute vectorial components j_e whose value does not exceed $r\omega_з^2$. Considering $r < 7000$ km, we obtain

$$j_e < 7 \cdot 10^6 (7.3 \cdot 10^{-5})^2 < 0.04 \text{ m/s}^2.$$

This means that centrifugal acceleration during calculation of the powered section can be disregarded. Finally, the value of Coriolis acceleration j_c does not exceed $2\omega_з v < 1.5 \cdot 10^{-4}$ v. It will be less than 0.05 m/s^2, if $v < (0.05)/(1.5 \cdot 10^{-4}) \approx 300$ m/s. The speed of the rocket takes considerably greater significance, but the component of speed $v_z = \dot z$ of the value 300 m/s usually is not reached. Therefore, members with ĉ $\dot z$ in expressions (1.11) for vector j_c in the calculation of the powered section can be disregarded, and we can use the following approximate expressions:

$$\left.\begin{aligned}
j_{cx} &= 2\omega_з \dot y \cos \varphi_ш \sin \psi = 2v\omega_з \cos \varphi_ш \sin \psi \sin \theta, \\
j_{cy} &= -2\omega_з \dot x \cos \varphi_ш \sin \psi = -2v\omega_з \cos \varphi_ш \sin \psi \cos \theta, \\
j_{cz} &= -2\omega_з \dot x \sin \varphi_ш + 2\omega_з \dot y \cos \varphi_ш \cos \psi = \\
&= 2v\omega_з(-\sin \varphi_ш \cos \theta + \cos \varphi_ш \cos \psi \sin \theta).
\end{aligned}\right\} \qquad (14.19)$$

Taking into account these remarks equation (14.17) takes the form

$$\frac{dv}{dt} = \frac{1}{m}(P - X_{1p} - c_x q S) - g \sin \theta - \frac{x}{r} g \cos \theta, \qquad (14.20)$$

since, on the basis of formulas (14.19),

$$j_{cx} \cos \theta + j_{cy} \sin \theta = 0.$$

In equation (14.18) we will produce analogous simplifications and will take out after the brackets the value $\varphi - \theta \approx A(\varphi_{пр} - \theta) = \alpha_y$. Let us then divide equation (14.18) by v and find:

$$\frac{d\theta}{dt} = \frac{1}{v}\left\{\frac{\alpha_y}{m}\left[P - X_{1p} + c'_{y_1} q S - 2a_u R' \frac{1 - A}{A}\right] - g \cos \theta + \frac{x}{r} g \sin \theta + 2v\omega_з \cos \varphi_ш \sin \psi\right\}.$$

or, taking into account expression (14.9),

$$\frac{d\theta}{dt}=\frac{1}{v}\left\{\frac{a_y}{m}\left[P-X_{1p}+\left(1-\frac{x_2-x_T}{l_1-x_T}\right)c'_{y_1}qS\right]-g\cos\theta+\frac{x}{r}g\sin\theta\right\}+2\omega_3\cos\varphi_e\sin\psi.$$

or, finally,

$$\frac{d\theta}{dt}=\frac{1}{v}\left[\frac{a_y}{m}\left(P-X_{1p}+\frac{l_1-x_2}{l_1-x_T}c'_{y_1}qS\right)-g\cos\theta+\frac{x}{r}g\sin\theta\right]+2\omega_3\cos\varphi_e\sin\psi. \qquad (14.21)$$

Equations (14.20) and (14.21) can be integrated jointly with the first two equations (13.3), since they will form a system of the first order equations with four unkown functions x, y, v and θ.

To these differential equations it is necessary to add the dependences (3.3), (10.20), (11.17), (14.9), (14.10), (6.16) and (14.4) for the determination of m, P, a_y, X_{1p}, where the value of drag of jet vanes can be considered constant; if, however, there are reliable characteristics of jet vanes, then it is possible to consider the dependence X_{1p} only from the deflection of control surfaces 2 and 4, disregarding the influence of very small angles of deflection of control surfaces 1 and 3 on the value of the total drag of the control surfaces.

The influence of small periodic oscillations of control surfaces on drag of control surfaces can be considered on the average. The mean value of the increase in vane drag from oscillations of control surfaces consists of half of the increase from the constant deflection of control surfaces by a value of the amplitude of oscillations.

After the motion of the rocket in plane Oxy is calculated, one can determine motion of the rocket in a lateral direction.

Differentiating the latter from equalities (13.3), we obtain

$$\ddot{z}=-\dot{v}\sigma-v\dot{\sigma},$$

or

$$v\frac{d\sigma}{dt}=-\ddot{z}-\sigma\frac{dv}{dt}.$$

Substituting $\ddot{z}$ and dv/dt from equations (14.14) and (14.20), we will find:

$$\begin{aligned}v\frac{d\sigma}{dt}={}&\frac{a_z}{m}\left[P-X_{1p}-c_xqS-\left(P-X_{1p}+c'_{y_1}qS\right)A+\right.\\&\left.+2a_0R'(1-A)\right]+\frac{z}{r}g-J_{ez}+J_{cz}-\\&-\frac{\sigma}{m}\left(P-X_{1p}-c_xqS\right)+\left(g\sin\theta+\frac{x}{r}g\cos\theta\right)\sigma.\end{aligned}$$

After transformations, the considering expression

$$a_z=-A\sigma,$$

emanating from equations (11.18) and (14.11), we will have

$$v\frac{d\sigma}{dt}=\frac{a_z}{m}\left[P-X_{1p}-c'_{y_1}qS-\frac{1-A}{A}2a_0R'\right]+\frac{z}{r}g+J_{ez}+J_{cz}+\left(g\sin\theta+\frac{x}{r}g\cos\theta\right)\sigma.$$

Considering the remarks made on the possible simplifications in last members, using expression (14.19) for J_{cz} and transforming the bracket just as with derivation of equation (14.21), we will obtain equation

$$\frac{d\sigma}{dt} = \frac{n_1}{mv}\left(P - \lambda_{1p} - \frac{l_1 - x_2}{l_1 - x_T} c'_{y_1} qS\right) + \frac{\sigma}{v} g \sin\theta -$$
$$- 2\omega_3(\sin\varphi_r \cos\theta - \cos\varphi_r \cos\psi \sin\theta), \qquad (14.22)$$

which together with

$$\frac{dz}{dt} = -v\sigma \qquad (14.23)$$

(see third equation of (11.4)) serves for calculation of lateral deflections.

Thus, the most general system of equations for ballistic calculations (14.20), (14.21), (13.3), (14.22) and (14.23) is obtained. This system can be used directly for numerical integration only when a number of constructive data of the rocket is known, namely:

a) accurate laws of the change in thrust P and flow rate per second $\dot{m}$ in flight;

b) accurate values of aerodynamic properties (c_x, c'_{y_1}, c_z) for different conditions of flight (M, h, α);

c) accurate characteristics of jet vanes (R', Q, λ);

d) parameters of the control system, in the first place, "program" of inclination of the axis of the rocket ($\varphi_{\text{пр}}$) and proportionality factor between the mean deviation of the axis of the rocket from the position prescribed to it by the control system and the mean deviation of the jet vanes (a_0).

Furthermore, it is assumed that calculation is produced for a definite position of the launch point and direction of firing (φ_r, ψ).

In the first stages of designing of the rocket enumerated constructive data are known only very approximately or are quite unknown. Their more precise definition is possible only on the basis of a number of laboratory, static and flight tests of the rocket and its units. These tests and the entire designing of rocket as a whole should be based in turn on preliminary calculations of trajectories, which are fulfilled on the basis of more or less simplified equations of motion. In these preliminary calculations of trajectories, which are fulfilled on the basis of more or less simplified equations of motion. In these preliminary calculations there usually is no special interest in the influence of the rotation of earth on the trajectory, since the main problem is the determination of mean values of flying characteristics of the rocket.

Our immediate problem will be the composition of simplified equations of motion which would correspond to some presence of initial constructive data and degree of their accuracy. Rotation of earth will not be considered, knowing that when necessary it can be considered by introduction of member $2\omega_3 \cos\varphi_r \sin\psi$ into the equation for $d\theta/dt$ and member $-2\omega_3(\sin\varphi_r \cos\theta - \cos\varphi_r \cos\psi \sin\theta)$ into the equation for $d\sigma/dt$. Then equation (14.22) at initial conditions $\xi = \sigma = 0$ when $t = 0$ gives for the whole powered section $\sigma = 0$, and therefore henceforth we will not write equations for σ and z, implying that motion is accomplished in plane Oxy. The angle α_z turns into zero, and angle α_y coincides with the angle of attack α. Subsequently we will use designation α instead of α_y.

Comparative calculations of trajectories and certain theoretical considerations show that a change in the form of the powered flight trajectory, i.e., a change in the dependence θ on t, has comparatively little influence on the speed of the rocket at the time of the turning off of the engine on the full flying range. Therefore, when the main problem is determination of distance, simplification of equations of motion can be allowed mainly owing to the equation for $d\theta/dt$.

In particular, if there are no data on the magnitude of coefficient a_0

characterizing the sensitivity of the control system of the rocket, it is possible to consider control system of "ideal," which corresponds to an infinite value of the coefficient a_0. Passing to the limit in the equation (14.8) when $a_0 \to \infty$, we will obtain

$$\alpha = \varphi - \theta = \varphi_{\text{пр}} - \theta. \tag{14.24}$$

and equations of motion take the form

$$\left.\begin{aligned}
\frac{dv}{dt} &= \frac{1}{m}(P - X_{1p} - c_x qS) - g\sin\theta - \frac{x}{r} g\cos\theta, \\
\frac{d\theta}{dt} &= \frac{1}{v}\left[\frac{\varphi_{\text{пр}} - \theta}{m}\left(P - X_{1p} + \frac{l_1 - x_{\text{д}}}{l_1 - x_{\text{т}}} c'_{y_1} qS\right) - \right. \\
&\qquad \left. - g\cos\theta + \frac{x}{r} g\sin\theta\right], \\
\frac{dx}{dt} &= v\cos\theta, \\
\frac{dy}{dt} &= v\sin\theta.
\end{aligned}\right| \tag{14.25}$$

If it is necessary to determine the program angle of the deflection of jet vanes δ_2, then it is possible, by using the last of relations (14.1), to obtain the expression

$$\delta_2 = \frac{c'_{y_1} qS(x_{\text{д}} - x_{\text{т}})}{2K'(l_1 - x_{\text{т}})}(\varphi - \theta). \tag{14.26}$$

If now one were to consider that when $a_0 \to \infty$ in the limit is found to be $\varphi = \varphi_{\text{пр}}$, as this ensues from relation (14.24), then expression (14.26), will take the following form:

$$\delta_2 = -\frac{c'_{y_1} qS(x_{\text{д}} - x_{\text{т}})}{2K'(l_1 - x_{\text{т}})}(\varphi_{\text{пр}} - \theta). \tag{14.27}$$

The system (14.25) is most commonly used in those cases when there is produced a checking calculation of the trajectory for the purpose of determining the parameters of motion of the rocket and loads having an effect on rocket on the powered section.

If the form of the trajectory, i.e., the dependence of the angle of inclination of tangent θ on the time of flight is assigned beforehand, then in system (14.25) one should jointly integrate only the first and the last two equations with unknown functions v, x and y. The second equation of this system can serve for the determination of the angle of attack

$$\alpha = \frac{m\left(v\frac{d\theta}{dt} + g\cos\theta - \frac{x}{r} g\sin\theta\right)}{P - X_{1p} + \frac{l_1 - x_{\text{д}}}{l_1 - x_{\text{т}}} c'_{y_1} qS}. \tag{14.28}$$

The angle of inclination of the axis of the rocket is determined with this by the formula

$$\varphi = \theta + \alpha. \tag{14.29}$$

The absence of exact values of aerodynamic coefficients and centering of the rocket is indicated greatest of all in the determination of the angle of inclination of tangent θ from the equation for $d\theta/dt$, in which the member depending on $l_1 - x_{\text{д}}$, $l_1 - x_{\text{т}}$ and c'_{y_1} is the main one in value. It is natural that the angle of attack α is determined inaccurately. Since the angle of attack of long-range rockets in flight is usually small, it is impossible, disregarding it, to calculate the trajectory by equations obtained from (14.25) when $\theta = \varphi_{\text{пр}}$ (in this case it is possible to disregard the member $\frac{x}{r} g$):

$$\left.\begin{aligned} \frac{dv}{dt} &= \frac{1}{m}(P - X_{1p} - c_x qS) - g\sin\varphi_{\text{пр}}, \\ \frac{dx}{dt} &= v\cos\varphi_{\text{пр}}, \\ \frac{dy}{dt} &= v\sin\varphi_{\text{пр}}. \end{aligned}\right\} \qquad (14.30)$$

The error practically appearing in the acceleration of rocket is less than the error due to the inaccuracy of 10-20% in value of the coefficient of drag.

The system (14.30) should be used for all design calculations, for further simplification of equations of motion, for appraisal of the influence on the motion of the rocket of different factors little connected with the form of trajectory, and in other cases not requiring special accuracy.

§ 15. Equations of Motion for the Region of Free Flight

In the solution of the basic problem of ballistics it is assumed that the rocket accomplishes free flight with an angle of attack equal to zero. It follows from this first that on the rocket act only two forces of all those examined by us: gravity and drag. Secondly, there is not need for equations of motion about the center of gravity. Consequently, the equations of motion (13.1) for the section of free flight take the form

$$\left.\begin{aligned} \ddot{x} &= -\frac{1}{m} c_x qS\cos\theta - \frac{x}{r} g - j_{ex} - j_{cx}, \\ \ddot{y} &= -\frac{1}{m} c_x qS\sin\theta - \frac{R+y}{r} g - j_{ey} - j_{cy}, \\ \ddot{z} &= \frac{1}{m} c_x qS\sigma - \frac{z}{r} g - j_{ez} - j_{cz}. \end{aligned}\right\} \qquad (15.1)$$

where v, θ and σ are determined by relation (13.3). The latter can be rewritten thus:

$$\left.\begin{aligned} \cos\theta &= \frac{v_x}{v}, \\ \sin\theta &= \frac{v_y}{v}, \\ \sigma &= -\frac{v_z}{v}. \end{aligned}\right\} \qquad (15.2)$$

Considering that $q = (\rho v^2/2)$, and using expressions (1.9)-(1.11), we will record equations of motion for the section of free flight in the following form:

$$\begin{aligned}\frac{dv_x}{dt} = -\frac{S}{2m}c_x\rho v v_x - \frac{g}{r}x - \omega_3^2\{[x\cos\varphi_r\cos\psi + {}\\ + (R+y)\sin\varphi_r - z\cos\varphi_r\sin\psi]\cos\varphi_r\cos\psi - x\} - {}\\ - 2\omega_3(v_y\cos\varphi_r\sin\psi + v_z\sin\varphi_r),\\ \frac{dv_y}{dt} = -\frac{S}{2m}c_x\rho v v_y - \frac{g}{r}(R+y) - \omega_3^2\{[x\cos\varphi_r\cos\psi + {}\\ + (R+y)\sin\varphi_r - z\cos\varphi_r\sin\psi]\sin\varphi_r - (R+y)\} + {}\\ + 2\omega_3(v_x\cos\varphi_r\sin\psi + v_z\cos\varphi_r\cos\psi),\\ \frac{dv_z}{dt} = -\frac{S}{2m}c_x\rho v v_z - \frac{g}{r}z - \omega_3^2\{-[x\cos\varphi_r\cos\psi + {}\\ + (R+y)\sin\varphi_r - z\cos\varphi_r\sin\psi]\cos\varphi_r\sin\psi - z\} + {}\\ + 2\omega_3(v_x\sin\varphi_r - v_y\cos\varphi_r\cos\psi),\end{aligned}$$

or finally in the following form which is more convenient for calculations by it:

$$\left.\begin{aligned}\frac{dv_x}{dt} = -kc_x\frac{\rho}{\rho_0}vv_x - \frac{g}{r}x + a_{11}x + a_{12}(R+y) + {}\\ + a_{13}z + b_{12}v_y + b_{13}v_z,\\ \frac{dv_y}{dt} = -kc_x\frac{\rho}{\rho_0}vv_y - \frac{g}{r}(R+y) + a_{21}x + {}\\ + a_{22}(R+y) + a_{23}z + b_{21}v_x + b_{23}v_z,\\ \frac{dv_z}{dt} = -kc_x\frac{\rho}{\rho_0}vv_z - \frac{g}{r}z + c_{31}x + a_{32}(R+y) + {}\\ + a_{33}z + b_{31}v_x + b_{32}v_y,\end{aligned}\right\} \qquad (15.3)$$

where

$$k = \frac{S\rho_0}{2m}; \qquad (15.4)$$

$$\left.\begin{aligned}a_{11} &= \omega_3^2(\sin^2\varphi_r + \cos^2\varphi_r\sin^2\psi),\\ a_{12} &= a_{21} = -\omega_3^2\sin\varphi_r\cos\varphi_r\cos\psi,\\ a_{13} &= a_{31} = \omega_3^2\cos^2\varphi_r\sin\psi\cos\psi,\\ a_{22} &= \omega_3^2\cos^2\varphi_r,\\ a_{23} &= a_{32} = \omega_3^2\sin\varphi_r\cos\varphi_r\sin\psi,\\ a_{33} &= \omega_3^2(\sin^2\varphi_r + \cos^2\varphi_r\cos^2\psi);\end{aligned}\right\} \qquad (15.5)$$

$$\left.\begin{aligned}b_{12} &= -b_{21} = -2\omega_3\cos\varphi_r\sin\psi,\\ b_{13} &= -b_{31} = -2\omega_3\sin\varphi_r,\\ b_{23} &= -b_{32} = 2\omega_3\cos\varphi_r\cos\psi\end{aligned}\right\} \qquad (15.6)$$

are constant for the given trajectory coefficients.

Equations (15.3), together with equations

$$\left.\begin{aligned}\frac{dx}{dt} &= v_x,\\ \frac{dy}{dt} &= v_y,\\ \frac{dz}{dt} &= v_z\end{aligned}\right\} \qquad (15.7)$$

should be applied in those cases when it is necessary to calculate the trajectory with great accuracy, for example, in the compilation of preliminary tables of firing for flight tests of a rocket.

It is necessary to consider that in the determination of flying range the accuracy of calculation of the section of free flight has much greater importance than the accuracy of calculation of the powered section. Actually, the deflection from the calculated trajectory which the rocket has at the end of the powered section to a greater or lesser degree is compensated by the control system, namely, the instrument, sluthing off the engine. Tuning of this instrument is produced by proceeding from calculation of the trajectory of free flight. Thus, the main factors affecting coincidence of calculation and actual range are the perfection of the control system and accuracy of calculation of the section of free flight.

On the other hand, possibilities of accurate calculation of the trajectory for free flight are considerably greater than those for the powered section, since a larger part of free flight lied in so rarefied layers of the atmosphere that the only force subject to calculation in equations of motion is gravity, which is well-known with great accuracy. If one were to consider the flatness of earth, one can determine the acceleration of gravity g with an accuracy of the order of 0.0004%. The assumption made by us on the sphericity of earth considerably lowers the accuracy with which the acceleration of gravity is determined, which is clear from Table 15.1. In this table R denotes the mean radius of earth and R', the true distance of the point from the center of earth.

Table 15.1

φ_r	g_0	$\frac{fM}{R^2}$	$\frac{fM}{R^2}-g_0$	$\frac{fM}{R'^2}$	$\frac{fM}{R'^2}-g_0$
0°	9,814	9,820	0,006	9,798	−0,016
30°	9,819	9,820	0,001	9,815	−0,004
60°	9,828	9,820	−0,008	9,848	0,020
90°	9,832	9,820	−0,012	9,864	0,032

Hence it is clear that formula

$$g = \frac{fM}{r^2} \qquad (15.8)$$

possesses an accuracy of the order of 0.01 m/s^2, or 0.1%. Furthermore, Table 15.1 shows that it is expedient quantity r to calculate by the formula

$$r = R + h,$$

where h is the actual altitude of the point above the surface of earth. If, however, by r we mean the true distance of the point from the center of earth, then the accuracy of formula (15.8) will be reduced almost three times.

Thus the accuracy of the right sides of the system (15.3) does not exceed 0.01 m/s^2, and we have the right to disregard in them members smaller than this value.

Since the value of coefficients a_{1k} does not exceed ω_3^2, members of the form $a_{11}x$ and $a_{13}z$ can be disregarded when $|x|$ or $|z| < (0.01/\omega_3^2) = 2 \cdot 10^6$ m = 2000 km. Members with coefficients b_{1k} can be disregarded, if the absolute value of the corresponding component of speed does not exceed

$$\frac{0{,}01}{\max|b_{ik}|} = \frac{0{,}01}{2\omega_3} \approx 10 \text{ m/s}.$$

This condition for long-range rockets can satisfy only by the component of speed v_z.

In the member $\frac{g}{r}x$ it is possible to replace r by R, if

$$g\left|\frac{x}{r}-\frac{x}{R}\right| = \frac{g\,|xh|}{rR} < 0{,}01 \text{ m/s}^2,$$

i.e., when

$$|xh| < \frac{0{,}01Rr}{g}$$

and all the more when

$$|xh| < \frac{0{,}01R^2}{g} \approx 40\,000 \text{ km}^2.$$

which takes place about the whole trajectory for firing ranges up to 500 km. Analogously one can assume that

$$\frac{g}{r}z = \frac{g}{R}z$$

when

$$|zh| < 40\,000 \text{ km}^2.$$

In all members containing **r**, it is possible to replace it by R + y, if $\frac{R+y}{r} > 0.999$ (proceeding from an accuracy of 0.1%), and this is fulfilled if

$$\begin{aligned}(R+y)^2 &> 0{,}998r^2,\\ x^2 &< 0{,}002r^2,\\ x &< 0{,}045r\end{aligned}$$

and all the more if $x < 0.045R \approx 280$ km.

Let us now turn to the simplification of equations for the section of free flight. Above it was already said that for calculation of average flying characteristics of the rocket the rotation of earth can be disregarded. With this it is better not simply to reject members considering the rotation of earth but replace them by mean values. At first we will calculate the mean values of coefficients a_{ik} and b_{ik} for the arbitrary point of launch on the surface of earth, changing the azimuth of firing ψ from 0 to 2π. These mean values of coefficients will be noted by primes. They are calculated by the formulas

$$a'_{ik} = \frac{1}{2\pi}\int_0^{2\pi} a_{ik}\,d\psi, \qquad i,\ k = 1,\ 2,\ 3.$$

Carrying beyond the integral sign the ψ factors not dependent on, we see that the calculation of coefficients is reduced to the calculation of integrals

$$\int_0^{2\pi} d\psi = 2\pi,$$

$$\int_0^{2\pi} \sin\psi \, d\psi = \int_0^{2\pi} \cos\psi \, d\psi = 0,$$

$$\int_0^{2\pi} \sin^2\psi \, d\psi = \int_0^{2\pi} \cos^2\psi \, d\psi = \pi,$$

$$\int_0^{2\pi} \sin\psi \cos\psi \, d\psi = 0.$$

Thus,

$$\left.\begin{aligned} a'_{11} &= \omega_3^2\left(\sin^2\varphi_r + \frac{1}{2}\cos^2\varphi_r\right), \\ a'_{22} &= \omega_3^2 \cos^2\varphi_r, \\ a'_{33} &= \omega_3^2\left(\sin^2\varphi_r + \frac{1}{2}\cos^2\varphi_r\right), \\ a'_{12} &= a'_{13} = a'_{21} = a'_{23} = a'_{31} = a'_{32} = 0; \end{aligned}\right\} \tag{15.9}$$

$$\left.\begin{aligned} -b'_{13} &= b'_{31} = 2\omega_3 \sin\varphi_r, \\ b'_{12} &= b'_{21} = b'_{23} = b'_{32} = 0. \end{aligned}\right\} \tag{15.10}$$

Now let us find the mean values a''_{ik} and b''_{ik} of coefficients a'_{ik} and b'_{ik} (there will be at the same time the mean values of coefficients a_{ik} and b_{ik}) over the entire surface of the earth S. These mean values are equal to

$$a''_{ik} = \frac{\int\int_S a'_{ik} \, dS}{S} = \frac{\int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} a'_{ik} \cos\varphi \, d\varphi}{\int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi} = \frac{2\pi \int_{-\pi/2}^{\pi/2} a'_{ik} \cos\varphi \, d\varphi}{4\pi},$$

or, finally,

$$a''_{ik} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} a'_{ik} \cos\varphi \, d\varphi.$$

Calculating the definite integrals

$$\int_{-\pi/2}^{\pi/2} \sin^2\varphi \cos\varphi \, d\varphi = \frac{2}{3},$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi \, d\varphi = \frac{4}{3},$$

$$\int_{-\pi/2}^{\pi/2} \sin\varphi \cos\varphi \, d\varphi = 0.$$

We find values of coefficients interesting to us

$$\left.\begin{aligned}
&a^{*}_{11}=a^{*}_{22}=a^{*}_{33}=\frac{2}{3}\omega_3^2=3.545\cdot 10^{-9}\,\frac{1}{s^{2}}.\\
&a^{*}_{12}=a^{*}_{13}=a^{*}_{21}=a^{*}_{23}=a^{*}_{31}=a^{*}_{32}=0.\\
&b^{*}_{12}=b^{*}_{13}=b^{*}_{21}=b^{*}_{23}=b^{*}_{31}=b^{*}_{32}=0.
\end{aligned}\right\}\qquad(15.11)$$

Consequently, the mean flying range of the rocket can be determined by integration of the system

$$\left.\begin{aligned}
&\frac{dv_x}{dt}=-kc_x\frac{\rho}{\rho_0}vv_x-\left(\frac{g}{r}-\frac{2}{3}\omega_3^2\right)x.\\
&\frac{dv_y}{dt}=-kc_x\frac{\rho}{\rho_0}vv_y-\left(\frac{g}{r}-\frac{2}{3}\omega_3^2\right)(R+y).\\
&\frac{dv_z}{dt}=-kc_x\frac{\rho}{\rho_0}vv_z-\left(\frac{g}{r}-\frac{2}{3}\omega_3^2\right)z
\end{aligned}\right\}\qquad(15.12)$$

together with equations (15.7).

The last equations of systems (15.7) and (15.12) show that at initial conditions $z = 0$ and $v_z = 0$ these equalities will be fulfilled along the entire trajectory, i.e., the mean trajectory of the [illegible]et lies in plane Oxy. Thus, the need in equations for z and v_z is eliminated.

System (15.12) is somewhat simplified during transition to polar coordinates, i.e., with replacement of coordinates by the formulas

$$\left.\begin{aligned}
x&=r\sin\chi,\\
R+y&=r\cos\chi,
\end{aligned}\right\}\qquad(15.13)$$

where χ is the central angle formed by rays drawn from the center of earth to the point of launch and to the rocket, in other words, the angle between the radius vectors of the launch point and rocket. Differentiating relations (15.13), we find:

$$\left.\begin{aligned}
\dot{x}&=\dot{r}\sin\chi+r\dot{\chi}\cos\chi.\\
\dot{y}&=\dot{r}\cos\chi-r\dot{\chi}\sin\chi;
\end{aligned}\right\}\qquad(15.14)$$

$$\left.\begin{aligned}
\ddot{x}&=\ddot{r}\sin\chi+2\dot{r}\dot{\chi}\cos\chi-r\dot{\chi}^2\sin\chi+r\ddot{\chi}\cos\chi.\\
\ddot{y}&=\ddot{r}\cos\chi-2\dot{r}\dot{\chi}\sin\chi-r\dot{\chi}^2\cos\chi-r\ddot{\chi}\sin\chi.
\end{aligned}\right\}\qquad(15.15)$$

From equations (15.13)-(15.15) the following relations ensue:

$$\left.\begin{aligned}
x\sin\chi+(R+y)\cos\chi&=r.\\
x\cos\chi-(R+y)\sin\chi&=0;
\end{aligned}\right\}\qquad(15.16)$$

$$\left.\begin{aligned}
\dot{x}\sin\chi+\dot{y}\cos\chi&=\dot{r}.\\
\dot{x}\cos\chi-\dot{y}\sin\chi&=r\dot{\chi};
\end{aligned}\right\}\qquad(15.17)$$

$$\left.\begin{aligned}
\ddot{x}\sin\chi+\ddot{y}\cos\chi&=\ddot{r}-r\dot{\chi}^2.\\
\ddot{x}\cos\chi-\ddot{y}\sin\chi&=2\dot{r}\dot{\chi}+r\ddot{\chi}.
\end{aligned}\right\}\qquad(15.18)$$

Substituting in relations (15.18) $\ddot{x}$ and $\ddot{y}$ from equations (15.12) and using relations (15.16) and (15.17), we will obtain a system of equations of motion in polar coordinates:

$$\left.\begin{aligned}\ddot{r} - r\dot{\chi}^2 &= -kc_x\frac{\rho}{\rho_0}v\dot{r} - g + \frac{2}{3}\omega_3^2 r,\\ r\ddot{\chi} + 2\dot{r}\dot{\chi} &= -kc_x\frac{\rho}{\rho_0}vr\dot{\chi}.\end{aligned}\right\} \qquad (15.19)$$

Speed v, on the basis of equations (15.14), is expressed by r and χ thus:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + (r\dot{\chi})^2. \qquad (15.20)$$

Altitude h and flying range along the arc of the earth's surface yield expressions

$$h = r - R, \qquad (15.21)$$

$$l = R\chi. \qquad (15.22)$$

Finally, if the altitude of flight is so great that drag can be disregarded, system (15.19) results in the form

$$\left.\begin{aligned}\ddot{r} - r\dot{\chi}^2 &= -g + \frac{2}{3}\omega_3^2 r,\\ r\ddot{\chi} + 2\dot{r}\dot{\chi} &= 0,\end{aligned}\right\} \qquad (15.23)$$

or, taking into account expression (15.8),

$$\left.\begin{aligned}\ddot{r} - r\dot{\chi}^2 &= -\frac{fM}{r^2} + \frac{2}{3}\omega_3^2 r,\\ \frac{d}{dt}(r^2\dot{\chi}) &= 0.\end{aligned}\right\} \qquad (15.24)$$

The second equation of (15.24) is written on the basis of equality

$$r(r\ddot{\chi} + 2\dot{r}\dot{\chi}) = \frac{d}{dt}(r^2\dot{\chi}). \qquad (15.25)$$

If in the first of equations (15.24) we disregard the member $\frac{2}{3}\omega_3^2 r$, which will lead, obviously, to an insignificant decrease in the calculation range as compared to the true, then we will obtain the system of equations

$$\left.\begin{aligned}\ddot{r} - r\dot{\chi}^2 &= -\frac{fM}{r^2},\\ \frac{d}{dt}(r^2\dot{\chi}) &= 0,\end{aligned}\right\} \qquad (15.26)$$

which is easily integrable in general form. We will arrive at this system if we examine not the relative but the absolute motion of the rocket at high altitudes, i.e., the motion in the inertial system of coordinates the origin of which moves together with the center of the earth, and the axes maintain a constant direction in space.

The absolute motion of the rocket is composed of the rotation of the earth and relative motion of the rocket. With motion at high altitude where aerodynamic forces are negligible, it is simpler to determine the relative motion not directly with the help of equations (15.3) but using the absolute motion which is described by simple equations (15.26). These questions are analyzed in Chapter V in greater detail.

§ 16. More Precise Definition of Equations of Motion

In the preceding paragraphs in speaking of the terrestrial system of coordinates, we did not distinguish two possible positions of this system. In § 1, for example, it was implied that axis Oy is directed along the radius of the earth, and in other places, in particular, in § 2 and 6, it was assumed that at the time of launch the direction of axis Oy coincides with the direction of the longitudinal axis of the rocket, i.e., with the direction of the plumb line. In the first assumption plane Oxz touches the surface of the sphere with the center coinciding with the center of earth, but in the second one it is the tangent plane to the actual surface of the terrestrial spheroid.

Deviation of the plumb line from the radius of the earth is caused, first, by the fact that on the body quiescent at the surface of the earth, besides the attractive force mg, there acts a centrifugal force $m\omega_3^2R\cos\varphi_r$, directed on the perpendicular to the axis of rotation of the earth. The resultant these forces mg' is gravity, and its direction is the direction of the plumb line. As can be seen from Fig. 16.1, this direction will form with the direction of the attractive force the angle γ', the value of which can be easily determined by the theorem of sines

$$\frac{\sin\gamma'}{m\omega_3^2R\cos\varphi_r}=\frac{\sin\varphi_r}{mg}.$$

The centrifugal force is small as compared to gravity (not greater than 0.35% of the latter), and therefore angle γ' is small and with sufficient accuracy one can assume that it is equal to

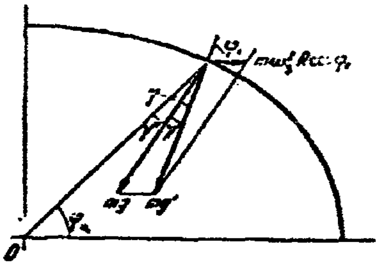

Fig. 16.1.

$$\gamma'=\frac{\omega_3^2R}{g}\sin\varphi_r\cos\varphi_r=\frac{\omega_3^2R}{2g}\sin 2\varphi_r. \qquad (16.1)$$

The maximum value of angle γ' consist of (for latitude 45°)

$$\gamma'_{\max}=\frac{\omega_3^2R}{2g}=0.00173=6'.$$

Secondly, the very direction of the attractive force does not coincide with the radius of the earth and due to the deviation of the form of earth from a sphere will form with the radius of earth (more accurately, with a straight line connecting the given point with the center of earth) the angle γ''. As a result the plumb line is deflected from the radius of earth at angle

$$\gamma=\gamma'+\gamma''. \qquad (16.2)$$

The value of angle γ is easily determined from the condition that the plumb line is normal to the surface of the terrestrial spheroid. If one were to assume the latter as a ellipsoid with a flattening $\alpha=(1/298.3)$ (Krasovskiy's ellipsoid), then for angle γ there can be obtained an expression accurate to values of the order of α^2,

$$\gamma=\alpha\sin 2\varphi_r. \qquad (16.3)$$

Angle γ has at latitude 45° a maximum value equal to

$$\gamma_{\max}=\alpha=0.00335=11'.5.$$

Thus, at the surface of the earth the centrifugal force caused by the rotation of the earth gives the same effect as that of flatness of the earth.

Let us keep the designation Oxyz for the system of coordinates rigidly joined with the earth, for which the origin is located at the launch point, axis Oy is directed upwards along the vertical directly opposite gravity, and axis Ox, lying just as axis Oz in a horizontal plane, will form with the plane of the prime meridian the angle ψ. This angle is called azimuthal aiming. Such a system of coordinates will be called launch, since at launch the connected axes of the rocket are oriented along axes of the launch system, namely, the O_1x_1 axis is combined with the Oyaxis, O_1y_1 axis is directed aside, directly opposite to Ox axis, and O_1z_1 axis is directed in parallel with axis Oz. Consequently, the axis of the gyro-instruments at launch are oriented along axes of the starting system: The axis of rotation of the gyroscope of gyro horizon is along the Ox axis, and the axis of rotation of the vertical gyro is along the Oz axis.

The second inaccuracy which was allowed up till now consists in the affirmation that the axis of rotation of gyroscopes of gyro-instruments remain in parallel with axes of the terrestrial system of coordinates. In reality they remain in parallel with the directions which had corresponding axes of the launch system at the time of launch. Axes of the launch system, being rigidly joined with earth, turn at time t at angle $\omega_3 t$ about the axis of rotation of the earth. For one minute the angle $\omega_3 t$ reaches a magnitude of 15'.

The position of axes of the launch system at the time of launch will be designated by the index "0." Thus we will use three systems of coordinates: bound $O_1x_1y_1z_1$, launch Oxyz and initial launch $Ox_0y_0z_0$.

Everything said in § 6 remains correct if instead of the terrestrial system we use the initial launch system of coordinates. In particular, angles φ, ξ and η should also be counted off with respect to the initial launch system. Consequently, Table 2.1 of direction cosines is correct for axes of the bound system of coordinates with respect to the initial launch system (Table 16.1).

Subsequently everywhere we will disregard the squares and pairwise products of angles γ, $\omega_3 t$ and other small angles. It is easy to verify that the direction cosines between axes of the initial launch and launch systems of coordinates will be determined by Table 16.2.

Table 16.1

	Ox_0	Oy_0	Oz_0
O_1x_1	$\cos\varphi$	$\sin\varphi$	$-\xi$
O_1y_1	$-\sin\varphi$	$\cos\varphi$	η
O_1z_1	$\xi\cos\varphi+\eta\sin\varphi$	$\xi\sin\varphi-\eta\cos\varphi$	1

Table 16.2

	Ox	Oy	Oz
Ox_0	1	$\omega_3 t\cos\varphi_r\sin\psi$	$\omega_3 t\sin\varphi_r$
Oy_0	$-\omega_3 t\cos\varphi_r\sin\psi$	1	$-\omega_3 t\cos\varphi_r\cos\psi$
Oz_0	$-\omega_3 t\sin\varphi_r$	$\omega_3 t\cos\varphi_r\cos\psi$	1

Let us introduce designations for the small angles appearing in Table 16.2:

$$\left.\begin{aligned}\gamma_1&=\omega_3 t\cos\varphi_r\cos\psi,\\ \gamma_2&=\omega_3 t\sin\varphi_r,\\ \gamma_2&=-\omega_3 t\cos\varphi_r\sin\psi.\end{aligned}\right\}\qquad(16.4)$$

With these designations Table 16.2 will take the following form (Table 16.3):

Table 16.3

	Ox	Oy	Oz
Ox_0	1	$-\gamma_3$	γ_2
Oy_0	γ_3	1	$-\gamma_1$
Oz_0	$-\gamma_2$	γ_1	1

Comparing Table 16.1 and 16.3 (multiplying matrices), we will obtain values of direction cosines between axes of the launch and bound systems of coordinates (Table 16.4). Here, on the basis of the smallness of angle γ_3 it was accepted that

$$\sin\gamma_3 = \gamma_3, \quad \cos\gamma_3 = 1$$

and, consequently,

$$\cos(\varphi - \gamma_3) = \cos\varphi + \gamma_3\sin\varphi,$$

$$\sin(\varphi - \gamma_3) = \sin\varphi - \gamma_3\cos\varphi.$$

Table 16.4

	Ox	Oy	Oz
O_1x_1	$\cos(\varphi - \gamma_3)$	$\sin(\varphi - \gamma_3)$	$\gamma_2\cos\varphi - \gamma_1\sin\varphi - \xi$
O_1y_1	$-\sin(\varphi - \gamma_3)$	$\cos(\varphi - \gamma_3)$	$-\gamma_1\cos\varphi - \gamma_2\sin\varphi + \eta$
O_1z_1	$\xi\cos\varphi + \eta\sin\varphi - \gamma_2$	$\xi\sin\varphi - \eta\cos\varphi + \gamma_1$	1

As was already noted, deviation of the plumb line from the radius of earth is caused almost in equal degree by the centrifugal force, conditioned by the rotation of earth, and the flatness of earth. Consequently, considering this deviation and determining its value by the formula (16.3), we are obliged at the same time to consider the deflection of the acceleration of the earth's gravity from the law expressed by formula (15.8).

It is known that the acceleration of terrestrial gravity, with an accuracy of values of the order of oblateness of the earth α, can be decomposed into two components:

radial

$$g_r = \frac{fM}{r^2} - \frac{\mu}{r^4}(3\sin^2\varphi_{\text{ц}} - 1)$$

and meridional

$$g_m = \frac{2\mu}{r^4}\sin\varphi_{\text{ц}}\cos\varphi_{\text{ц}},$$

where r is the distance of the examined point from the center of earth; φ_{II} – the geocentric latitude of the point; $fM = 3.9862 \times 10^{14}$ m^3/s^2 – the product of the gravitational constant by the mass of earth; $\mu = fMa^2(\alpha - \frac{m}{2}) = 26.245 \cdot 10^{24}$ m^5/s^2; $m = (\omega_3^2a^3)/(fM)$ – the ratio of centrifugal acceleration to the acceleration of

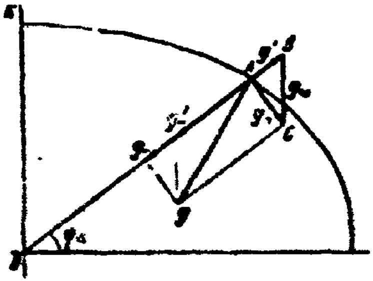

Fig. 16.2.

gravity at the equator.

The radial component g_r is directed to the center of earth, and the meridional g_m is perpendicular to it and lies in a plane of the meridian and is directed in the direction of the equator (Fig. 16.2).

It is expediently to decompose vector g_m into two (non-orthogonal), components g' and g_ω in a direction of radius OA and axis of rotation of earth ON. From the triangle ABC there can be obtained for these components expressions

$$\left.\begin{aligned} g' &= g_m \operatorname{tg}\varphi_a = \frac{2\mu}{r^4}\sin^2\varphi_a, \\ g_\omega &= g_m\frac{1}{\cos\varphi_a} = \frac{2\mu}{r^4}\sin\varphi_a. \end{aligned}\right\} \tag{16.5}$$

Component g' acts in a direction directly opposite the direction of the radial acceleration g_r. These two components of acceleration can be united in one expression

$$g_r' = \frac{fM}{r^2} - \frac{\mu}{r^4}(5\sin^2\varphi_a - 1). \tag{16.6}$$

Thus, in the composition of equations of motion, taking into account the flatness of earth, it is necessary to consider two components of the acceleration of terrestrial gravity: g_r', directed to the center of earth and g_ω, directed in parallel to the axis of rotation of earth. We will again deduce equations of motion, repeating the way already done in § 11-14.

As a basis will use the launch system of coordinates. Angles θ and σ will determine the direction of the tangent to the trajectory in this system. Consequently formulas (11.1) and (11.2) and all of them ensuing, including (13.3) and (14.16), will remain in force.

Direction cosines of tractive force and forces from controls in the launch system of coordinates can be found from Table 16.4, since these forces act along axes of the bound system of coordinates. Two component forces of terrestrial gravity have direction cosines represented in Table 16.5. In this table x_c, y_c, z_c are coordinates of the center of earth in the launch system of coordinates. They can be calculated by the formulas

$$\left.\begin{aligned} x_c &= r_0\sin\gamma\cos\psi, \\ y_c &= -r_0\cos\gamma, \\ z_c &= -r_0\sin\gamma\sin\psi. \end{aligned}\right\} \tag{16.7}$$

where r_0 is the distance of the launch point from the center of earth. With an error having an order of α^2, the following formula are correct:

$$r_0 = a(1 - \alpha\sin^2\varphi_r); \tag{16.8}$$

$$\left.\begin{aligned} x_c &= a\alpha\sin 2\varphi_r\cos\psi, \\ y_c &= -a(1 - \alpha\sin^2\varphi_r), \\ z_c &= -a\alpha\sin 2\varphi_r\sin\psi. \end{aligned}\right\} \tag{16.9}$$

The value of component force of terrestrial gravity appearing in Table 16.5 is determined with the help of formulas (16.5) and (16.6). Entering into these formulas,

Table 16.5

	Ox	Oy	Oz
g'_r	$-\frac{x-x_c}{r}$	$-\frac{y-y_c}{r}$	$-\frac{z-z_c}{r}$
g_ω	$-\frac{\omega_{3x}}{\omega_3}$	$-\frac{\omega_{3y}}{\omega_3}$	$-\frac{\omega_{3z}}{\omega_3}$

the quantities r and $\varphi_{\text{Ш}}$ are determined by the formulas

$$r = \sqrt{(x-x_c)^2+(y-y_c)^2+(z-z_c)^2} \tag{16.10}$$

and

$$\sin\varphi_{\text{ш}} = r^0\cdot\omega_3^0 = \frac{(x-x_c)\omega_{3x}+(y-y_c)\omega_{3y}+(z-z_c)\omega_{3z}}{r\omega_3} = \frac{r_\omega}{r}, \tag{16.11}$$

where r_ω is the projection of the radius vector r on the axis of rotation of earth expressed by formula

$$r_\omega = \frac{(x-x_c)\omega_{3x}+(y-y_c)\omega_{3y}+(z-z_c)\omega_{3z}}{\omega_3}. \tag{16.12}$$

Formulas for the determination of vectorial components of angular velocity of the rotation of earth on axes of the launch system of coordinates do not differ from those derived in the first part:

$$\left.\begin{aligned} \omega_{3x} &= \omega_3\cos\varphi_r\cos\psi,\\ \omega_{3y} &= \omega_3\sin\varphi_r,\\ \omega_{3z} &= -\omega_3\cos\varphi_r\sin\psi. \end{aligned}\right| \tag{16.13}$$

For the determination of aerodynamic forces and moments it is possible, as before, to use vector formulas (11.7)-(11.10) and formulas (11.19) and (11.20) ensuing from them, where in the latter

$$\begin{aligned} a_y &= -v^0\cdot y_1^0 = \cos\theta\sin(\varphi-\gamma_3)-\sin\theta\cos(\varphi-\gamma_3)+\\ &\quad+\sigma(-\gamma_1\cos\varphi-\gamma_2\sin\varphi+\eta)\approx\\ &\quad\approx\sin(\varphi-\gamma_3-\theta)\approx\varphi-\gamma_3-\theta,\\ a_z &= v^0\cdot z_1^0 = \cos\theta(\xi\cos\varphi+\eta\sin\varphi-\gamma_2)+\\ &\quad+\sin\theta(\xi\sin\varphi-\eta\cos\varphi+\gamma_1)-\sigma\approx\\ &\quad\approx\xi\cos(\varphi-\theta)+\eta\sin(\varphi-\theta)+\gamma_1\sin\theta-\gamma_2\cos\theta-\sigma. \end{aligned}$$

The last formula, since angle $\varphi - \theta$ is small, can be rewritten in the form

$$a_z = \xi-\sigma+\gamma_1\sin\theta-\gamma_2\cos\theta.$$

Direction cosines of moments M_{ay_1} and M_{az_1} coincide with components of vectors y_1^0 and z_1^0 (Table 16.4) and direction cosines of forces Y_y and Y_z, with direction cosines of vectors

$$-v^0 \times z_1^0 =$$

$$= -\begin{vmatrix} x^0 & y^0 & z^0 \\ \cos\theta & \sin\theta & -\sigma \\ \xi\cos\varphi + \eta\sin\varphi - \gamma_2 & \xi\sin\varphi - \eta\cos\varphi + \gamma_1 & 1 \end{vmatrix} =$$

$$= -\sin\theta x^0 + \cos\theta y^0 + (\eta - \gamma_1\cos\theta - \gamma_2\sin\theta) z^0$$

and

$$-v^0 \times y_1^0 =$$

$$= -\begin{vmatrix} x^0 & y^0 & z^0 \\ \cos\theta & \sin\theta & -\sigma \\ -\sin(\varphi-\gamma_3) & \cos(\varphi-\gamma_3) & -\gamma_1\cos\varphi - \gamma_2\sin\varphi + \eta \end{vmatrix} =$$

$$= -[(\eta - \gamma_1\cos\varphi - \gamma_2\sin\varphi)\sin\theta + \sigma\cos(\varphi-\gamma_3)] x^0 +$$

$$+ [(\eta - \gamma_1\cos\varphi - \gamma_2\sin\varphi)\cos\theta - \sigma\sin(\varphi-\gamma_3)] y^0 -$$

$$- \cos(\varphi - \gamma_3 - \theta) z^0.$$

Here it is possible to assume $\cos(\varphi - \gamma_3 - \theta) = \cos\alpha_y = 1$. Both vecotrs $v^0 \times z_1^0$ and $v^0 \times y_1^0$, just as in § 11, can be considered unity.

Now it is again possible to copy tables of forces and moments having an effect on the rocket (Tables 16.6 and 16.7), where damping moments will not be considered. Hence with those same assumptions as in §§ 13 and 14, we will obtain the following equations of motion:

$$m(\ddot{x} + j_{ex} + j_{cx}) = -m\left[\frac{g_r'}{r}(x - x_c) + \frac{g_\omega}{\omega_3}\omega_{3x}\right] -$$

$$- c_x qS\cos\theta - c_y' qS(\varphi - \gamma_3 - \theta)\sin\theta +$$

$$+ (P - X_{1p})\cos(\varphi - \gamma_3) - 2R'\delta_2\sin(\varphi - \gamma_3).$$

$$m(\ddot{y} + j_{ey} + j_{cy}) = -m\left[\frac{g_r'}{r}(y - y_c) + \frac{g_\omega}{\omega_3}\omega_{3y}\right] -$$

$$- c_x qS\sin\theta + c_y' qS(\varphi - \gamma_3 - \theta)\cos\theta +$$

$$+ (P - X_{1p})\sin(\varphi - \gamma_3) + 2R'\delta_2\cos(\varphi - \gamma_3).$$

$$m(\ddot{z} + j_{ez} + j_{cz}) = -m\left[\frac{g_r'}{r}(z - z_c) + \frac{g_\omega}{\omega_3}\omega_{3z}\right] +$$

$$+ c_x qS\sigma - c_y' qS(\zeta - \sigma + \gamma_1\sin\theta - \gamma_2\cos\theta) -$$

$$- (P - X_{1p})(\zeta + \gamma_1\sin\varphi - \gamma_2\cos\varphi) + R'(\delta_1 + \delta_2).$$

$$R'\lambda_1(\delta_2 - \delta_1) = 0.$$

$$-c_y' qS(x_d - x_T)(\zeta - \sigma + \gamma_1\sin\theta - \gamma_2\cos\theta) +$$

$$+ R'(l_1 - x_T)(\delta_1 + \delta_2) = 0.$$

$$-c_y' qS(x_d - x_T)(\varphi - \gamma_3 - \theta) - 2R'(l_1 - x_T)\delta_2 = 0.$$

Table 16.6

Force	Quantity	Cosine of angle with axis		
		Ox	Oy	Oz
Gravity	$G_r = mg'_r$	$-\frac{x-x_c}{r}$	$-\frac{y-y_c}{r}$	$-\frac{z-z_c}{r}$
	$G_\omega = mg_\omega$	$-\omega_{3x}/\omega_3$	$-\omega_{3y}/\omega_3$	$-\omega_{3z}/\omega_3$
Drag	$X = c_x qS$	$-\cos\theta$	$-\sin\theta$	σ
Lift	$Y_y = c'_{y_1} qS\alpha_y$	$-\sin\theta$	$\cos\theta$	$\eta-\gamma_1\cos\theta-\gamma_2\sin\theta$
Lift	$Y_z = c'_{y_1} qS\alpha_z$	$-(\eta-\gamma_1\cos\varphi-\gamma_2\sin\varphi)\times$ $\times\sin\theta+\sigma\cos(\varphi-\gamma_3)$	$(\eta-\gamma_1\cos\varphi-\gamma_2\sin\varphi)\times$ $\times\cos\theta-\sigma\sin(\varphi-\gamma_3)$	-1
Axial force from jet vanes	$X_{1p} = 4Q_{p0} +$ $+\lambda(\delta_1^2+2\delta_2^2+\delta_3^2)$	$-\cos(\varphi-\gamma_3)$	$-\sin(\varphi-\gamma_3)$	$\xi+\gamma_1\sin\varphi-\gamma_2\cos\varphi$
Lateral force from jet vanes 2 and 4	$Y_{1p} = 2R'\delta_2$	$-\sin(\varphi-\gamma_3)$	$\cos(\varphi-\gamma_3)$	$\eta-\gamma_1\cos\varphi-\gamma_2\sin\varphi$
Lateral force from jet vanes 1 and 3	$Z_{1p} = R'(\delta_1+\delta_3)$	$\xi\cos\varphi+\eta\sin\varphi-\gamma_2$	$\xi\sin\varphi-\eta\cos\varphi+\gamma_1$	1
Thrust	$P = \frac{\dot{m}}{m_e}(P_e+S_a p_a)-$ $-S_a p$	$\cos(\varphi-\gamma_3)$	$\sin(\varphi-\gamma_3)$	$-\xi-\gamma_1\sin\varphi+$ $+\gamma_2\cos\varphi$

Table 16.7

Moment	Quantity	Cosine of angle with axis		
		O_1x_1	O_1y_1	O_1z_1
Aerodynamic moment	$M_{ay_1} = -c'_{y_1}qS(x_d-x_T)\alpha_z$	0	1	0
Aerodynamic moment	$M_{az_1} = -c'_{y_1}qS(x_d-x_T)\alpha_y$	0	0	1
Moment of jet vanes 1 and 3 with respect to axis O_1x_1	$M_{x_1} = R'h_1(\delta_3-\delta_1)$	1	0	0
Moment of jet vanes 1 and 3 with respect to axis O_1y_1	$M_{y_1} = R'(l_1-x_T)(\delta_1+\delta_3)$	0	1	0
Moment of jet vanes 2 and 4 with respect to axis O_1z_1	$M_{z_1} = -2R'(l_1-x_T)\delta_2$	0	0	1

Formula (14.4) for angles of deflection of control surfaces remain in force, and using them, we obtain:

$$\left.\begin{aligned}
\ddot{x} = \frac{1}{m}\Big[(P - X_{1p})\cos(\varphi - \gamma_3) - c_x qS\cos\theta - \\
- c'_{y_1} qS(\varphi - \gamma_3 - \theta)\sin\theta - 2a_0 R' \Delta\varphi \sin(\varphi - \gamma_3)\Big] - \\
- \frac{g'_r}{r}(x - x_c) - \frac{g_\omega}{\omega_3}\omega_{3x} - j_{ex} - j_{cx}, \\
\ddot{y} = \frac{1}{m}\Big[(P - X_{1p})\sin(\varphi - \gamma_3) - c_x qS\sin\theta + \\
+ c'_{y_1} qS(\varphi - \gamma_3 - \theta)\cos\theta + 2a_0 R' \Delta\varphi \cos(\varphi - \gamma_3)\Big] - \\
- \frac{g'_r}{r}(y - y_c) - \frac{g_\omega}{\omega_3}\omega_{3y} - j_{ey} - j_{cy}, \\
\ddot{z} = -\frac{1}{m}\Big[(P - X_{1p} + c'_{y_1} qS)(\zeta + \gamma_1\sin\varphi - \gamma_2\cos\varphi) - \\
- (c'_{y_1} + c_x) qS\sigma + 2a_0 R'\dot{\zeta}\Big] - \\
- \frac{g'_r}{r}(z - z_c) - \frac{g_\omega}{\omega_3}\omega_{3z} - j_{ez} - j_{cz}
\end{aligned}\right| \qquad (16.14)$$

and

$$\eta = 0,$$

$$c'_{y_1} qS(x_a - x_T)(\zeta - \sigma + \gamma_1\sin\theta - \gamma_2\cos\theta) + \\ + 2a_0 K'(l_1 - x_T)\dot{\zeta} = 0,$$

$$c'_{y_1} qS(x_a - x_T)(\varphi - \gamma_3 - \theta) + 2a_0 R'(l_1 - x_T)\Delta\varphi = 0.$$

Simplifying these equations in the same way as in deriving dependences (14.10)-(14.13), we obtain:

$$\left.\begin{aligned}
&\eta = 0, \\
&\alpha_z = -A(\sigma - \gamma_1\sin\theta + \gamma_2\cos\theta), \\
&\zeta = (1 - A)(\sigma - \gamma_1\sin\theta + \gamma_2\cos\theta);
\end{aligned}\right\} \qquad (16.15)$$

$$\left.\begin{aligned}
&\alpha_y = A(\varphi_{np} - \gamma_3 - \theta), \\
&\Delta\varphi = -(1 - A)(\varphi_{np} - \gamma_3 - \theta),
\end{aligned}\right\} \qquad (16.16)$$

where by A, as before, we designate the quantity

$$A = \frac{2a_0 R'(l_1 - x_T)}{2a_0 R'(l_1 - x_T) + c'_{y_1} qS(x_a - x_T)}. \qquad (16.17)$$

The equations obtained differ from equations §§ 13 and 14 by the presence of members containing the coefficient μ (in components of the acceleration of gravity), and members depending on angles γ_1, γ_2 and γ_3 (in equations of motion about the center of gravity). It is not always necessary to consider these members. The group of members containing γ_1, γ_2 and γ_3, characterizes the behavior of gyroscopes with respect to the system of coordinates revolving together with the earth. Calculation of these members gives during calculation of the powered section approximately the same effect as that of the calculation of Coriolis acceleration, and should be produced in all those cases when the latter is taken into account. During calculation of the section of free flight this group of members, naturally, loses meaning.

Calculation in equations of motion of members dependent of μ gives a result of one order, taking into account centrifugal acceleration. During calculation of the powered section and section of free flight for distances of the order of several hundreds of kilometers these members, along with centrifugal acceleration, can be disregarded. Such an assumption is equivalent to the fact that the field of gravity is assumed central and purely Newtonian: the acceleration of gravity is found to be inversely proportional to the square of the distance from the attracting center located on axis Oy.

After these remarks we will write the equation of motion for the powered section and for the section of free flight.

For the powered section in the launch system of coordinates we obtain the following equation with the help of the same transformations as in § 14:

$$\left.\begin{aligned}
&\frac{dv}{dt}=\frac{1}{m}(P-X_{1p}-c_x qS)-g\sin\theta-\frac{x}{r}g\cos\theta,\\
&\frac{d\theta}{dt}=\frac{1}{v}\left\{\frac{\alpha_y}{m}\left[P-X_{1p}+\frac{(l_1-x_2)c'_{y_1}}{l_1-x_T}qS\right]-\right.\\
&\qquad\left.-g\cos\theta+\frac{x}{r}g\sin\theta\right\}+2\omega_3\cos\varphi_r\sin\psi,\\
&\frac{d\sigma}{dt}=\frac{1}{v}\left\{\frac{\alpha_z}{m}\left[P-X_{1p}+\frac{(l_1-x_2)c'_{y_1}}{l_1-x_T}qS\right]+\right.\\
&\qquad\left.+g\sigma\sin\theta\right\}-2\omega_3(\sin\varphi_r\cos\theta-\\
&\qquad\qquad-\cos\varphi_r\cos\psi\sin\theta),\\
&\frac{dx}{dt}=v\cos\theta,\\
&\frac{dy}{dt}=v\sin\theta,\\
&\frac{dz}{dt}=-v\sigma,
\end{aligned}\right\}\qquad(16.18)$$

where for the determination of angles α_y and α_z, the relations (16.15)-(16.17) serve, and angles γ_1, γ_2, γ_3 are determined by the formulas (16.4). The acceleration of gravity g is calculated by the formula

$$g=g_0\frac{R^2}{r^2}.\qquad(16.19)$$

It is expedient to determine the acceleration of gravity at the surface of earth g_0 depending upon the latitude of the launch point by the formula

$$g_0=9.7805+0.0519\sin^2\varphi_r.\qquad(16.20)$$

Equations of motion for the section of free flight in the launch system of coordinates with those same assumptions (for small flying ranges) have the form

$$\left.\begin{aligned}
&\frac{dv_x}{dt}=-kc_x\frac{\rho}{\rho_0}vv_x-\frac{g}{r}x+b_{12}v_y+b_{13}v_z,\\
&\frac{dv_y}{dt}=-kc_x\frac{\rho}{\rho_0}vv_y-\frac{g}{r}(R+y)+b_{21}v_x+b_{23}v_z,\\
&\frac{dv_z}{dt}=-kc_x\frac{\rho}{\rho_0}vv_z-\frac{g}{r}z+b_{31}v_x+b_{32}v_y,\\
&\frac{dx}{dt}=v_x,\\
&\frac{dy}{dt}=v_y,\\
&\frac{dz}{dt}=v_z.
\end{aligned}\right|\qquad(16.21)$$

where coefficients k and b_{ik} are expressed by formulas (15.4) and (15.6), and g is determined with the help of formulas (16.19) and (16.20). It is recommended to use these equations for distances of the order 500 km and below. For greater distances one should use the equation of motion composed in the launch system of coordinates, taking into account the noncentralness of the field of gravity. We will obtain them from equations (16.14) in the same way, as were derived equations (15.3) from equations (13.1):

$$\left.\begin{aligned}
\frac{dv_x}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_x - \frac{g_r'}{r}(x-x_c) - \frac{g_\omega}{\omega_3}\omega_{3x} + \\
&\quad + a_{11}(x-x_c) + a_{12}(y-y_c) + \\
&\qquad + a_{13}(z-z_c) + b_{12}v_y + b_{13}v_z,\\
\frac{dv_y}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_y - \frac{g_r'}{r}(y-y_c) - \frac{g_\omega}{\omega_3}\omega_{3y} + \\
&\quad + a_{21}(x-x_c) + a_{22}(y-y_c) + \\
&\qquad + a_{23}(z-z_c) + b_{21}v_x + b_{23}v_z,\\
\frac{dv_z}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_z - \frac{g_r'}{r}(z-z_c) - \frac{g_\omega}{\omega_3}\omega_{3z} + \\
&\quad + a_{31}(x-x_c) + a_{32}(y-y_c) + \\
&\qquad + a_{33}(z-z_c) + b_{31}v_x + b_{32}v_y,\\
\frac{dx}{dt} &= v_x,\\
\frac{dy}{dt} &= v_y,\\
\frac{dz}{dt} &= v_z,
\end{aligned}\right\} \qquad (16.22)$$

where

$$k = \frac{S\rho_0}{2m}; \qquad (16.23)$$

$$\left.\begin{aligned}
&a_{11} = \omega_3^2 - \omega_{3x}^2, && a_{23} = a_{32} = -\omega_{3y}\omega_{3z}, && b_{23} = -b_{32} = 2\omega_{3x},\\
&a_{22} = \omega_3^2 - \omega_{3y}^2, && a_{31} = a_{13} = -\omega_{3x}\omega_{3z}, && b_{31} = -b_{13} = 2\omega_{3y},\\
&a_{33} = \omega_3^2 - \omega_{3z}^2, && a_{12} = a_{21} = -\omega_{3x}\omega_{3y}, && b_{12} = -b_{21} = 2\omega_{3z};
\end{aligned}\right\} \qquad (16.24)$$

quantities g_r' and g_ω are determined by the formulas (16.6) and (16.5), and quantities ω_{3x}, ω_{3y} and ω_{3z} by the formulas (15.13).

For the determination of the altitude of the point of trajectory above the survace of the earth, it is possible to use formula

$$h = r - r_3,$$

where r_3 is the radius vector of points of the surface of the terrestrial spheroid for a geocentric latitude $\varphi_{\text{ц}}$; it is determined by the formula

$$r_3 = a(1 - \alpha\sin^2\varphi_{\text{ц}}).$$

Calculation of the section of free flight is conducted according to encounter of the rocket with the surface of the earth, i.e., prior to h = 0.

The flying range and azimuth of the launch line to the impact point can be determined with the help of geodesic tables. It is also possible to recommend the following method of approximation. The central angle β is calculated by the formula

$$\cos\beta = \frac{(x_0 - x_c)(x_{\text{п}} - x_c) + (y_0 - y_c)(y_{\text{п}} - y_c) + (z_0 - z_c)(z_{\text{п}} - z_c)}{r_0 r_{\text{п}}} \quad (16.25)$$

(the index "п" pertains to the impact point), and the flying range is determined by the approximate formula

$$L = \frac{r_0 + r_{\text{п}}}{2}\beta. \quad (16.26)$$

The angular deviation of the impact point from the plane of aiming is calculated by the formula

$$\operatorname{tg}\Delta\psi_{\text{п}} = \frac{z_{\text{п}}}{x_{\text{п}}}. \quad (16.27)$$

CHAPTER V

THEORY OF FREE FLIGHT AT HIGH ALTITUDES

§ 17. Absolute and Relative Motion

Motion of the rocket on the section of free flight, on the assumption that the angle of attack all the time is equal to zero, occurs under the action of two forces: gravity and drag. If one were to examine the motion only at high altitudes where the drag is practically equal to zero, then gravity remains the only force subject to calculation in equations of motion.

In this chapter again we will consider earth as a sphere, the field of gravity central, and the acceleration of gravity variable inversely proportional to the square of the distance from the center of earth:

$$g = g_0 \frac{R^2}{r^2}.$$

It is found profitable to use the inertial system of coordinates O'x'y'z' (Fig. 17.1), moving forward, evenly and rectilinearly together with the center of the earth where we place the origin of coordinated O'. Equations of motion in such system have the form (§ 15)

$$\left.\begin{aligned} \ddot{x}' &= -\frac{x'}{r} g, \\ \ddot{y}' &= -\frac{y'}{r} g, \\ \ddot{z}' &= -\frac{z'}{r} g. \end{aligned}\right| \tag{17.1}$$

If we arrange the system O'x'y'z' in such a manner that at the initial moment of motion the plane O'x'y' passes through the radius vector of rocket r and through the vector of absolute velocity v' (i.e., velocity in the system O'x'y'z' and not speed v relative to the terrestrial system), then we will have initial conditions

$$z' = \dot{z}' = 0, \tag{17.2}$$

and the third equation (17.1) shows that equality (17.2) will be fulfilled during the entire flight, i.e., the whole trajectory will lie in plane O'x'y'.

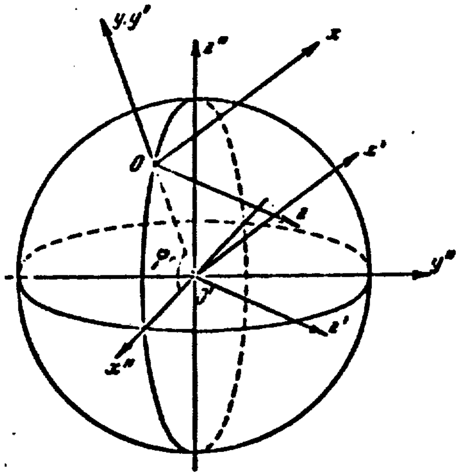

Fig. 17.1

Introducing in this plane the polar coordinates

$$x' = r\cos\chi, \quad y' = r\sin\chi,$$

and producing the same transformations in § 15, we obtain the equations, of motion

$$\left.\begin{aligned} &\ddot{r} - r\dot{\chi}^2 = -\frac{fM}{r^2}, \\ &\frac{d}{dt}(r^2\dot{\chi}) = 0, \end{aligned}\right\} \tag{17.3}$$

coinciding with equations (15.26). Let us remember that equations (15.26) were obtained for the determination of the average flight trajectory (with different positions of the launch point and directions of firing) in the uninertial system of coordinates connected with earth. Hence beccmes clear the importance of the system (17.3) for calculation of both the absolute (in the system O'x'y'z'), and relative (in the terrestrial system Oxyz) motion of the rocket.

But the importance of this system is not exhausted by the determination of the flight path at high altitudes. If the problem consists only in the determination of flying range, then the system (17.3) can be used for the approximate calculation of the whole section of free flight, since the influence of drag of air on the form of the trajectory and on the full distance proves to be very small. It decreases with an increase in range and, consequently, the speed of flight.

So that the problem of absolute motion of the rocket becomes defined, it is necessary to find initial conditions for this motion, assuming initial conditions for the relative motion of the rocket, i.e., coordinates, of the rocket x_H, y_H, z_H, speed v_H and its direction determined angles θ_H and σ_H in the terrestrial system of coordinates Oxyz are well-known.

Let us introduce one more inertial system O'x"y"z", rigidly joined with the system O'x'y'z'. Let us dispose it in such a manner that at the initial moment of time (i.e., at the time t_H) the plane O'x"y" coincides with the plane of the equator and plane O'x"z" passes through the origin O of the terrestrial system of coordinates (Fig. 17.1). We will call the initial point for the absolute motion point H. Then at the time t_H coordinates of this point x''_H, y''_H, z''_H will be connected with coordinates x_H, y_H, z_H by formulas

$$\left.\begin{aligned}x''_H&=-x_H\sin\varphi_r\cos\psi+(R+y_H)\cos\varphi_r+z_H\sin\varphi_r\sin\psi,\\ y''_H&=x_H\sin\psi+z_H\cos\psi,\\ z''_H&=x_H\cos\varphi_r\cos\psi+(R+y_H)\sin\varphi_r-z_H\cos\varphi_r\sin\psi.\end{aligned}\right|\qquad(17.4)$$

Components of relative speed v_H about axes of the system O'x"y"z" are expressed in terms of $\dot{x}_H$, $\dot{y}_H$, and $\dot{z}_H$ by analogous formulas. Regarding the absolute velocity v'_H, it is composed of the speed v_H and velocity of following

$$\omega_3\times r_H=\omega_3 z''^0\times(x''_H x''^0+y''_H y''^0+z''_H z''^0)=$$
$$=\omega_3(x''_H y''^0-y''_H x''^0).$$

Thus components of absolute velocity will be

$$\left.\begin{aligned}\dot{x}''_H&=-\dot{x}_H\sin\varphi_r\cos\psi+\dot{y}_H\cos\varphi_r+\dot{z}_H\sin\varphi_r\sin\psi-\omega_3 y''_H,\\ \dot{y}''_H&=\dot{x}_H\sin\psi+\dot{z}_H\cos\psi+\omega_3 x''_H,\\ \dot{z}''_H&=\dot{x}_H\cos\varphi_r\cos\psi+\dot{y}_H\sin\varphi_r-\dot{z}_H\cos\varphi_r\sin\psi,\end{aligned}\right|\qquad(17.5)$$

where quantities $\dot{x}_H$, $\dot{y}_H$, $\dot{z}_H$ are determined by the formulas (13.3):

$$\left.\begin{aligned}\dot{x}_H&=v_H\cos\theta_H,\\ \dot{y}_H&=v_H\sin\theta_H,\\ \dot{z}_H&=-v_H\sigma_H.\end{aligned}\right|\qquad(17.6)$$

From quantities x''_H, y''_H, z''_H and $\dot{x}''_H$, $\dot{y}''_H$, $\dot{z}''_H$ it is easy to turn to geographic coordinates of point H.

Let us designate the geographic latitude of point H by $\varphi_{\Gamma H}$, the longitude by λ_H and longitude of launch point by λ_0. Then along with formulas (17.4) we will have (Fig. 17.2)

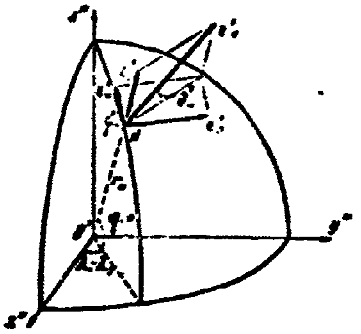

Fig. 17.2

$$x''_H=r_H\cos\varphi_{\Gamma H}\cos(\lambda_H-\lambda_0),$$
$$y''_H=r_H\cos\varphi_{\Gamma H}\sin(\lambda_H-\lambda_0),$$
$$z''_H=r_H\sin\varphi_{\Gamma H},$$

whence

$$\left.\begin{aligned}\operatorname{tg}(\lambda_H-\lambda_0)&=\frac{y''_H}{x''_H},\\ \operatorname{tg}\varphi_{\Gamma H}&=\frac{z''_H}{\sqrt{x''^2_H+y''^2_H}}=\frac{z''_H\cos(\lambda_H-\lambda_0)}{x''_H},\\ r_H&=\sqrt{x''^2_H+y''^2_H+z''^2_H}=\frac{z''_H}{\sin\varphi_{\Gamma H}}.\end{aligned}\right|\qquad(17.7)$$

Finally, we designate by θ'_H the angle formed by the vector of absolute velocity v'_H with the horizontal plane at point H, and by ψ'_H, the absolute azimuth, i.e., the angle which the horizontal component of absolute velocity will form with a direction to the north. Let us find components of absolute velocity in the direction of the meridian, parallels, and verticals, at point H, projecting directly on these directions vector v'_H and separately its components $\dot{x}''_H$, $\dot{y}''_H$ and $\dot{z}''_H$ (see Fig. 17.2):

$$\left.\begin{aligned}
v'_м &= v'_н\cos\theta'_н\cos\psi'_н = -\dot{x}''_н\sin\varphi_{гн}\cos(\lambda_н-\lambda_0)-\\
&\qquad -\dot{y}''_н\sin\varphi_{гн}\sin(\lambda_н-\lambda_0)+\dot{z}''_н\cos\varphi_{гн},\\
v'_п &= v'_н\cos\theta'_н\sin\psi'_н =\\
&\qquad = -\dot{x}''_н\sin(\lambda_н-\lambda_0)+\dot{y}''_н\cos(\lambda_н-\lambda_0),\\
v'_в &= v'_н\sin\theta'_н = \dot{x}''_н\cos\varphi_{гн}\cos(\lambda_н-\lambda_0)+\\
&\qquad +\dot{y}''_н\cos\varphi_{гн}\sin(\lambda_н-\lambda_0)+\dot{z}''_н\sin\varphi_{гн}.
\end{aligned}\right| \tag{17.8}$$

Hence it is already easy to find v'_H, ϑ'_H and ψ'_H by the formulas analogous to (17.7):

$$\left.\begin{aligned}
\operatorname{tg}\psi'_н &= \frac{v'_п}{v'_м},\\
\operatorname{tg}\theta'_н &= \frac{v'_в\cos\psi'_н}{v'_м} = \frac{v'_в\sin\psi'_н}{v'_п} = \frac{v'_в}{\sqrt{v'^2_м+v'^2_п}},\\
v'_н &= \frac{v'_в}{\sin\theta'_н} = \frac{\sqrt{v'^2_м+v'^2_п}}{\cos\theta'_н} = \sqrt{v'^2_м+v'^2_п+v'^2_в}.
\end{aligned}\right| \tag{17.9}$$

Formulas (17.4)-(17.9) determine initial conditions of the absolute motion of the rocket on the section of free flight.

Our most immediate problem is the integration of system (17.3). Results of integration will be directly applicable to absolute motion, but for relative motion they give only a mean trajectory for different directions of firing and launch points.

§ 18. Integration of Equations of Motion

We approach the integration of equations of motion of the rocket in a vacuum which have the form

$$\ddot{r} - r\dot{\chi}^2 = -\frac{fM}{r^2}, \tag{18.1}$$

$$\frac{d}{dt}(r^2\dot{\chi}) = 0, \tag{18.2}$$

where (Fig. 18.1) r – the polar radius vector of the center of gravity of the rocket with respect to the center of earth; χ – the angle between the radius vector r and some axis passing through the center of earth and accepted as origin of the reading of the angles χ; f – the gravitational constant; M – the mass of earth.

Let us denote $k = fM$. As was already said in § 3, quantity k is equal to

$$k = 3.9862\cdot 10^{14}\ \mathrm{m}^3/\mathrm{s}^2.$$

Integrating equation (18.2), we obtain

$$r^2\dot{\chi} = c_1. \qquad (18.3)$$

Let us write this expression thus:

$$r\left(r\frac{d\chi}{dt}\right) = c_1.$$

With this

$$r\frac{d\chi}{dt} = v_r,$$

where v_r is the horizontal component of speed at a given point of the trajectory. Designating by θ the angle of inclination of the velocity vector to the local horizon, we obtain

$$v_r = v\cos\theta,$$

where v is the speed along the trajectory.

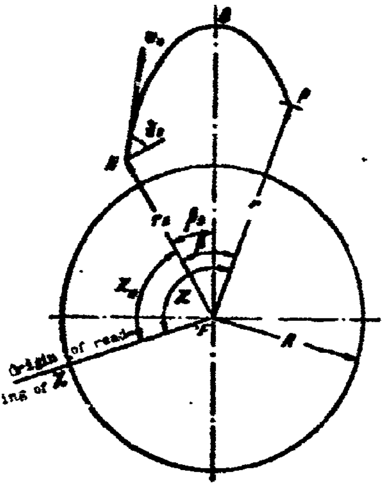

Fig. 18.1

Then formula (18.3) will be thus copied:

$$c_1 = rv\cos\theta. \qquad (18.4)$$

Value c_1 will be determined from initial conditions of the motion of the rocket

$$c_1 = r_k v_k \cos\theta_k. \qquad (18.5)$$

We copy equation (18.1) and (18.2) in such a form:

$$\ddot{r} - r\dot{\chi}^2 = -\frac{k}{r^2}, \qquad (18.6)$$

$$2r\dot{r}\dot{\chi} + r^2\ddot{\chi} = 0. \qquad (18.7)$$

Multiplying equation (18.6) by $2\dot{r}$, and equation (18.7) by $2\dot{\chi}$, and adding the results term by term, we get

$$2\dot{r}\ddot{r} - 2r\dot{r}\dot{\chi}^2 + 4r\dot{r}\dot{\chi}^2 + 2r^2\dot{\chi}\ddot{\chi} = -\frac{2k\dot{r}}{r^2}.$$

or

$$2(\dot{r}\ddot{r} + r\dot{r}\dot{\chi}^2 + r^2\dot{\chi}\ddot{\chi}) = -\frac{2k\dot{r}}{r^2}. \tag{18.8}$$

Differentiating expression (15.20) for speed

$$v^2 = \dot{r}^2 + r^2\dot{\chi}^2, \tag{18.9}$$

we have

$$\frac{d(v^2)}{dt} = 2(\dot{r}\ddot{r} + r\dot{r}\dot{\chi}^2 + r^2\dot{\chi}\ddot{\chi}). \tag{18.10}$$

Comparing expressions (18.8) and (18.10), we find

$$\frac{d(v^2)}{dt} = -\frac{2k}{r^2}\frac{dr}{dt} = \frac{d}{dt}\left(\frac{2k}{r}\right).$$

i.e.,

$$\frac{d}{dt}\left(v^2 - \frac{2k}{r}\right) = 0,$$

or

$$v^2 - \frac{2k}{r} = c_2. \tag{18.11}$$

This relation is correct for any moment of time, in particular, when $t = t_H$ we will obtain

$$c_2 = v_H^2 - \frac{2k}{r_H}. \tag{18.12}$$

From equations (18.9) and (18.11) we have

$$\dot{r}^2 + r^2\dot{\chi}^2 = \frac{2k}{r} + c_2,$$

whence

$$\dot{r} = \sqrt{c_2 + \frac{2k}{r} - r^2\dot{\chi}^2}.$$

considering that

$$\dot{r} = \frac{dr}{d\chi}\dot{\chi}$$

and (from equation (18.3))

$$\dot{\chi} = \frac{c_1}{r^2},$$

we will have

$$\dot{r} = \frac{dr}{d\chi}\cdot\frac{c_1}{r^2} = \sqrt{c_2 + \frac{2k}{r} - r^2\frac{c_1^2}{r^4}}.$$

whence

$$d\chi = \frac{\frac{c_1}{r^2}}{\sqrt{c_2 + \frac{2k}{r} - \frac{c_1^2}{r^2}}}dr. \tag{18.13}$$

Let us convert the denominator:

$$\sqrt{c_2+\frac{2k}{r}-\frac{c_1^2}{r^2}}=\sqrt{c_2+\frac{k^2}{c_1^2}-\left(\frac{c_1}{r}-\frac{k}{c_1}\right)^2}=$$

$$=\sqrt{c_2+\frac{k^2}{c_1^2}}\sqrt{1-\frac{\left(\frac{c_1}{r}-\frac{k}{c_1}\right)^2}{c_2+\frac{k^2}{c_1^2}}}. \qquad (18.14)$$

Let us show that the quantity under first radical is not negative:

$$c_2+\frac{k^2}{c_1^2}\geqslant 0.$$

Indeed, by inserting instead of c_1 and c_2 their expression (18.5) and (18.12), we will obtain

$$c_2+\frac{k^2}{c_1^2}=v_H^2-\frac{2k}{r_H}+\frac{k^2}{r_H^2v_H^2\cos^2\vartheta_H}=$$

$$=\left(v_H-\frac{k}{r_Hv_H}\right)^2+\left(\frac{k\,\mathrm{tg}\,\vartheta_H}{r_Hv_H}\right)^2\geqslant 0.$$

If $v_H^2 r_H \neq k$ or $\vartheta_H \neq 0$, then

$$c_2+\frac{k^2}{c_1^2}>0,$$

and we have the right to designate

$$\frac{\left(\frac{c_1}{r}-\frac{k}{c_1}\right)^2}{c_2+\frac{k^2}{c_1^2}}=u^2, \qquad (18.15)$$

whence

$$u=\frac{\frac{c_1}{r}-\frac{k}{c_1}}{\sqrt{c_2+\frac{k^2}{c_1^2}}}. \qquad (18.16)$$

Differentiating, we find

$$du=-\frac{c_1}{\sqrt{c_2+\frac{k^2}{c_1^2}}}\frac{dr}{r^2}. \qquad (18.17)$$

Considering expressions, (18.14), (18.15) and (18.17), we reduce equation (18.13) to the form

$$d\chi=-\frac{du}{\sqrt{1-u^2}}.$$

Integration gives

$$\chi=\arccos u+c_3,$$

or

$$u=\cos(\chi-c_3). \qquad (18.18)$$

Returning to the variable r, from expressions (18.16) and (18.18) we obtain

$$r=\frac{\frac{c_1^2}{k}}{1+\cos(\chi-c_3)\sqrt{1+c_2\frac{c_1^2}{k^2}}}.$$

Designating for brevity the constants

$$p=\frac{c_1^2}{k}, \qquad (18.19)$$

$$e=\sqrt{1+c_2\frac{c_1^2}{k^2}}, \qquad (18.20)$$

will find the equation of trajectory in the form

$$r=\frac{p}{1+e\cos(\chi-c_3)}. \qquad (18.21)$$

It is known that this is the equation of conic section in polar coordinates where p is the parameter of the section, and is its eccentricity.

Let us introduce angle (see Fig. 18.1)

$$\beta=\chi-\chi_H,$$

where χ_H is the angle characterizing the position of the initial radius vector r_H with respect to the axis accepted as the origin of reading of the angles, and β is the angle determining the position of the rocket at any moment of time relative to the initial radius vector. Consequently,

$$\chi=\chi_H+\beta. \qquad (18.22)$$

Substituting the expression (18.22) in equation (18.21), we obtain

$$r=\frac{p}{1+e\cos(\beta+\chi_H-c_3)}. \qquad (18.23)$$

Let us designate β_B as the angle corresponding to the peak of the trajectory. At the peak of the trajectory ($\beta=\beta_B$) r has a maximum value. From (18.23) it follows that this can be only if

$$\cos(\beta_B+\chi_H-c_3)=-1,$$

i.e., when

$$\beta_B+\chi_H-c_3=\pi.$$

Hence we have

$$\chi_H-c_3=\pi-\beta_B. \qquad (18.24)$$

Inserting (18.24) in (18.23), we finally obtain the equation of the trajectory

$$r=\frac{p}{1-e\cos(\beta_B-\beta)}. \qquad (18.25)$$

Since the form of the trajectory is characterized by the eccentricity, let us find the expression for e. Inserting into formula (18.20), instead of c_1 and c_2, their values (18.4) and (18.11), we obtain

$$e=\sqrt{1+\frac{\left(v^2-\frac{2k}{r}\right)r^2v^2\cos^2\theta}{k^2}}=$$

$$=\sqrt{1+\left(\frac{v^4r^2}{k^2}-\frac{2v^2r}{k}\right)\cos^2\theta}\,.$$

Introducing the new value

$$\nu=\frac{v^2r}{k}, \tag{18.26}$$

we find

$$e=\sqrt{1-(2-\nu)\nu\cos^2\theta}. \tag{18.27}$$

From initial data we have

$$\nu_H=\frac{v_H^2r_H}{k}, \tag{18.28}$$

$$e=\sqrt{1-(2-\nu_H)\nu_H\cos^2\theta_H}. \tag{18.29}$$

Formula (18.29) shows that the trajectory will be

elliptic when $\nu_H < 2$, since $e < 1$;
parabolic when $\nu_H = 2$, since $e = 1$;
hyperbolic when $\nu_H > 2$, since $e > 1$.

Thus the trajectory of the rocket in its absolute motion with respect to earth is a conic section one of the focuses of which is in the center of earth.

For a long time parabolic and hyperbolic trajectories, departing into infinity, were practical. For this reason the theory of motion of bodies in a vacuum under the action of the attraction of earth obtained in ballistics the name elliptic theory.

Subsequently we will examine only elliptic trajectories and use designations shown in Fig. 18.2; F — center of earth which is one of the focuses of the ellipse; O — launch; H — initial point of elliptic section of the trajectory; C — impact point of the rocket; x_H and y_H — coordinates of initial point with respect to the launch point; r_H — radius vector of initial point with respect to center of earth; v_H — speed at initial point; θ_H — angle of inclination of velocity vector at the initial point to the horizon; $R = r_C$ — radius of earth (radius vector of impact point); B — peak of the trajectory; δ — central angle corresponding to powered flight trajectory; β_C — central angle corresponding to section of free flight; β_B — central angle determining position of peak of trajectory relative to the initial radius vector.

We will consider that angle δ and the initial radius vector r_H (or the initial altitude $h_H = r_H - R$) are known, since they are easily determined from the assigned coordinates x_H and y_H of the initial point with respect to the launch place. Actually, from Fig. 18.2 we have

$$\operatorname{tg}\delta=\frac{x_H}{R+y_H} \tag{18.30}$$

and

$$r_{н} = \frac{R + y_{н}}{\cos\delta} = \frac{x_{н}}{\sin\delta}. \qquad (18.31)$$

For a solution of the practical problems the whole section of free flight will be taken as elliptic, including with respect to the little atmosphere the section during approach to the target. This will not introduce great error in the determination of the distance, since great drag of air on the atmospheric section almost does not change the elliptic form of the trajectory. It has considerable influence only on the speed and time of motion of the rocket.

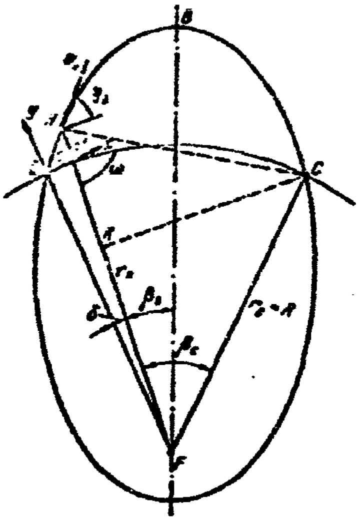

Fig. 18.2

Let us agree by flying range to mean the length of the arc on the surface of earth. Then full flying range from launch to the target will be equal to

$$L = l_{н} + l_{св}.$$

where l_H – distance of powered section; l_{CB} – distance of section of free flight. Subsequently we will consider that the distance of the powered section is already defined by the formula $l_H = R\delta$, where δ is from equation (18.30).

§ 19. Applications of Elliptic Theory

Let us enumerate six practically important problems:

1. To find distance from the assigned to speed, altitude and angle at the initial point.

2. From the assigned distance, altitude and angle find the necessary speed.

3. From the assigned distance and altitude find the optimum angle requiring the minimum speed.

4. From the assigned speed and altitude find the optimum angle providing the ultimate range.

5. To determine the change in distance depending upon small increases in altitude, speed and angle at the time of the turning off of the engine.

6. To determine parameters of motion about the trajectory.

Let us turn to detailed examination of the enumerated problems.

1. From the assigned to speed v_H, altitude h_H, and angle ϑ_H to find distance L.

From equation (18.25) we have

$$\cos(\beta_{н} - \beta) = \frac{1}{e}\left(1 - \frac{p}{r}\right). \qquad (19.1)$$

Using relations (18.4), (18.26) and (18.19), obtain

$$p = \frac{r^2 v^2 \cos^2\vartheta}{k} = \nu r \cos^2\vartheta. \qquad (19.2)$$

in particular,

$$p = \nu_{н} r_{н} \cos^2\vartheta_{н}. \qquad (19.3)$$

We substitute (19.1) by (19.2):

$$\cos(\beta_{\text{в}}-\beta)=\frac{1}{e}(1-\nu\cos^2\vartheta). \qquad (19.4)$$

It is obvious that

$$\sin(\beta_{\text{в}}-\beta)=\sqrt{1-\cos^2(\beta_{\text{в}}-\beta)}=\sqrt{1-\frac{1}{e^2}(1-\nu\cos^2\vartheta)^2}=$$
$$=\frac{1}{e}\sqrt{e^2-(1-\nu\cos^2\vartheta)^2}.$$

Considering expression (18.27), we will have

$$\sin(\beta_{\text{в}}-\beta)=\frac{1}{e}\sqrt{1-2\nu\cos^2\vartheta+\nu^2\cos^4\vartheta-1+2\nu\cos^2\vartheta-\nu^2\cos^4\vartheta},$$

i.e.,

$$\sin(\beta_{\text{в}}-\beta)=\frac{1}{e}\nu\sin\vartheta\cos\vartheta. \qquad (19.5)$$

With extraction of the root there is preserved only the plus sign, since on the ascenting phase $\beta < \beta_B$ and $\vartheta > 0$, and on descending $\beta > \beta_B$ and $\vartheta < 0$, i.e., $\sin(\beta_B - \beta)$ and $\sin\vartheta$ always have identical signs.

When $\beta = 0$ we obtain

$$\cos\beta_{\text{в}}=\frac{1}{e}(1-\nu_{\text{н}}\cos^2\vartheta_{\text{н}}), \qquad (19.6)$$

$$\sin\beta_{\text{в}}=\frac{1}{e}\nu_{\text{н}}\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}. \qquad (19.7)$$

Let us present expression (19.1) in the following form:

$$\cos\beta_{\text{в}}\cos\beta+\sin\beta_{\text{в}}\sin\beta=\frac{1}{e}\left(1-\frac{p}{r}\right).$$

Substituting instead of $\cos\beta_B$, $\sin\beta_B$ and p their expressions (19.6), (19.7) and (19.3), we will find

$$(1-\nu_{\text{н}}\cos^2\vartheta_{\text{н}})\cos\beta+\nu_{\text{н}}\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}\sin\beta=$$
$$=1-\frac{\nu_{\text{н}}r_{\text{н}}\cos^2\vartheta_{\text{н}}}{r}. \qquad (19.8)$$

Expressing $\sin\beta$ and $\cos\beta$ in terms of $\operatorname{tg}\frac{\beta}{2}$ according to formulas

$$\sin\beta=\frac{2\operatorname{tg}\frac{\beta}{2}}{1+\operatorname{tg}^2\frac{\beta}{2}},$$

$$\cos\beta=\frac{1-\operatorname{tg}^2\frac{\beta}{2}}{1+\operatorname{tg}^2\frac{\beta}{2}}$$

and multiplying both sides of the equation by $1 + \operatorname{tg}^2\frac{\beta}{2}$ we have

$$(1-\nu_{\text{н}}\cos^2\vartheta_{\text{н}})\left(1-\operatorname{tg}^2\frac{\beta}{2}\right)+2\nu_{\text{н}}\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}\operatorname{tg}\frac{\beta}{2}=$$
$$=\left(1-\frac{\nu_{\text{н}}r_{\text{н}}\cos^2\vartheta_{\text{н}}}{r}\right)\left(1+\operatorname{tg}^2\frac{\beta}{2}\right).$$

We obtained the quadratic equation with respect to tg $\frac{\beta}{2}$. Let us site in it similar members:

$$\left(2-\nu_{\text{н}}\cos^2\vartheta_{\text{н}}-\frac{\nu_{\text{н}}r_{\text{н}}\cos^2\vartheta_{\text{н}}}{r}\right)\operatorname{tg}^2\frac{\beta}{2}-$$
$$-2\nu_{\text{н}}\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}\operatorname{tg}\frac{\beta}{2}+\nu_{\text{н}}\cos^2\vartheta_{\text{н}}-\frac{\nu_{\text{н}}r_{\text{н}}\cos^2\vartheta_{\text{н}}}{r}=0,$$

or

$$[2r-(r_{\text{н}}+r)\nu_{\text{н}}\cos^2\vartheta_{\text{н}}]\operatorname{tg}^2\frac{\beta}{2}-$$
$$-2\nu_{\text{н}}r\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}\operatorname{tg}\frac{\beta}{2}-(r_{o}-r)\nu_{\text{н}}\cos^2\vartheta_{\text{н}}=0.$$

Dividing by $\cos^2\vartheta_H$, we obtain

$$[2r(1+\operatorname{tg}^2\vartheta_{\text{н}})-(r_{\text{н}}+r)\nu_{\text{н}}]\operatorname{tg}^2\frac{\beta}{2}-$$
$$-2\nu_{\text{н}}r\operatorname{tg}\vartheta_{\text{н}}\operatorname{tg}\frac{\beta}{2}-(r_{\text{н}}-r)\nu_{\text{н}}=0. \quad (19.9)$$

This equation, connecting the current values r and β with initial values r_H, ν_H, ϑ_H, essentially coincides with equation (18.25), but in form is considerably more convenient since the initial values enter into it in evident form. With its help it is easy to solve the majority of applied problems of elliptic theory set above.

We will designate for brevity the coefficients of this equation by

$$a=2r(1+\operatorname{tg}^2\vartheta_{\text{н}})-(r_{\text{н}}+r)\nu_{\text{н}}. \quad (19.10)$$

$$b=\nu_{\text{н}}r\operatorname{tg}\vartheta_{\text{н}}. \quad (19.11)$$

$$c=\nu_{\text{н}}(r_{\text{н}}-r). \quad (19.12)$$

The equation will take the form of

$$a\operatorname{tg}^2\frac{\beta}{2}-2b\operatorname{tg}\frac{\beta}{2}-c=0.$$

Solving this equation with respect to tg $\frac{\beta}{2}$, we find

$$\operatorname{tg}\frac{\beta}{2}=\frac{b\pm\sqrt{b^2+ac}}{a}, \quad (19.13)$$

where the minus sign corresponds to angles β of the ascending phase.

Consequently, to obtain the distance of the whole elliptic section it is necessary to assume r = R and in formula (19.13) to take the plus sign. Then we will obtain

$$\left.\begin{aligned}\operatorname{tg}\frac{\beta_C}{2}&=\frac{b+\sqrt{b^2+ac}}{a},\\ L_{co}&=R\beta_C.\end{aligned}\right\} \quad (19.14)$$

Thus we have the following diagram of calculation of distance:

$$\nu_{\text{н}} = \frac{v_{\text{н}}^2 r_{\text{н}}}{k},$$

$$a = 2R(1 + \operatorname{tg}^2 \theta_{\text{н}}) - (r_{\text{н}} + R)\nu_{\text{н}},$$

$$b = \nu_{\text{н}} R \operatorname{tg} \theta_{\text{н}},$$

$$c = \nu_{\text{н}}(r_{\text{н}} - R) = \nu_{\text{н}} h_{\text{н}},$$

$$\operatorname{tg} \frac{\beta_C}{2} = \frac{b + \sqrt{b^2 + ac}}{a},$$

$$l_{\text{сп}} = R\beta_C,$$

$$L = l_{\text{н}} + l_{\text{сп}}.$$

For the particular case r = r_H we have $\frac{\beta}{2} = \beta_B$ and from equations (19.10)-(19.12)

$$a = 2r_{\text{н}}(1 + \operatorname{tg}^2 \theta_{\text{н}} - \nu_{\text{н}}), \quad b = \nu_{\text{н}} r_{\text{н}} \operatorname{tg} \theta_{\text{н}}, \quad c = 0.$$

Consequently, from equation (19.13) we obtain

$$\operatorname{tg} \frac{\beta}{2} = \operatorname{tg} \beta_{\text{в}} = \frac{2b}{a},$$

i.e.,

$$\operatorname{tg} \beta_{\text{в}} = \frac{\nu_{\text{н}} \operatorname{tg} \theta_{\text{н}}}{1 + \operatorname{tg}^2 \theta_{\text{н}} - \nu_{\text{н}}} \tag{19.15}$$

and

$$l = 2R\beta_{\text{в}}.$$

2. To find the necessary speed v_H from the assigned distance L, angle θ_H, altitude h_H.

From formula (18.28) we have

$$v_{\text{н}} = \sqrt{\frac{k\nu_{\text{н}}}{r_{\text{н}}}}. \tag{19.16}$$

Consequently, it is necessary to find ν_H, which can be done by the following two methods:

First method. From equation (19.9) we have

$$\left[(r_{\text{н}} + r) \operatorname{tg}^2 \frac{\beta}{2} + 2r \operatorname{tg} \theta_{\text{н}} \operatorname{tg} \frac{\beta}{2} + r_{\text{н}} - r\right] \nu_{\text{н}} = 2r \operatorname{tg}^2 \frac{\beta}{2} (1 + \operatorname{tg}^2 \theta_{\text{н}});$$

hence

$$\nu_{\text{н}} = \frac{2r \operatorname{tg}^2 \frac{\beta}{2} (1 + \operatorname{tg}^2 \theta_{\text{н}})}{(r_{\text{н}} + r) \operatorname{tg}^2 \frac{\beta}{2} + 2r \operatorname{tg} \theta_{\text{н}} \operatorname{tg} \frac{\beta}{2} + r_{\text{н}} - r} \tag{19.17}$$

When r = R and β = β_C we obtain

$$\nu_{\text{н}}=\frac{2R\,\mathrm{tg}^2\frac{\beta_C}{2}\,(1+\mathrm{tg}^2\vartheta_{\text{н}})}{(r_{\text{н}}+R)\,\mathrm{tg}^2\frac{\beta_C}{2}+2R\,\mathrm{tg}\,\vartheta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}+r_{\text{н}}-R}.$$

Second method. From (19.8) we have (when r = r_C and β = β_C)

$$r_C\cos\beta_C-r_C\cos\beta_C\cos^2\vartheta_{\text{н}}\cdot\nu_{\text{н}}+$$
$$+r_C\sin\beta_C\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}\cdot\nu_{\text{н}}+r_{\text{н}}\cos^2\vartheta_{\text{н}}\cdot\nu_{\text{н}}=r_C,$$

i.e.,

$$\nu_{\text{н}}=\frac{r_C(1-\cos\beta_C)}{\cos\vartheta_{\text{н}}(r_{\text{н}}\cos\vartheta_{\text{н}}-r_C\cos\beta_C\cos\vartheta_{\text{н}}+r_C\sin\beta_C\sin\vartheta_{\text{н}})}=$$
$$=\mathrm{tg}\frac{\beta_C}{2}\,\frac{r_C\sin\beta_C}{\cos\vartheta_{\text{н}}[(r_{\text{н}}-r_C\cos\beta_C)\cos\vartheta_{\text{н}}+r_C\sin\beta_C\sin\vartheta_{\text{н}}]}=$$
$$=\mathrm{tg}\frac{\beta_C}{2}\,\frac{1}{\cos\vartheta_{\text{н}}\left(\frac{r_{\text{н}}-r_C\cos\beta_C}{r_C\sin\beta_C}\cos\vartheta_{\text{н}}+\sin\vartheta_{\text{н}}\right)}.$$

On Fig. 18.2 we draw segments HC and CK perpendicular to HF. Designating by ω angle FHC, we will have

$$\mathrm{tg}\,\omega=\frac{CK}{HK}=\frac{r_C\sin\beta_C}{r_{\text{н}}-r_C\cos\beta_C}. \tag{19.18}$$

Considering (19.18), we get

$$\nu_{\text{н}}=\mathrm{tg}\frac{\beta_C}{2}\,\frac{1}{\cos\vartheta_{\text{н}}\left(\frac{\cos\vartheta_{\text{н}}}{\mathrm{tg}\,\omega}+\sin\vartheta_{\text{н}}\right)}=$$
$$=\mathrm{tg}\frac{\beta_C}{2}\,\frac{\sin\omega}{\cos\vartheta_{\text{н}}(\cos\vartheta_{\text{н}}\cos\omega+\sin\vartheta_{\text{н}}\sin\omega)}$$

and, finally,

$$\nu_{\text{н}}=\mathrm{tg}\frac{\beta_C}{2}\,\frac{\sin\omega}{\cos\vartheta_{\text{н}}\cos(\omega-\vartheta_{\text{н}})}. \tag{19.19}$$

Thus we have the following two diagram of the calculation of speed. The first diagram of calculation:

$$\beta_C=\frac{L-l_{\text{н}}}{R}.$$
$$\nu_{\text{н}}=\frac{2R(1+\mathrm{tg}^2\vartheta_{\text{н}})\,\mathrm{tg}^2\frac{\beta_C}{2}}{(r_{\text{н}}+R)\,\mathrm{tg}^2\frac{\beta_C}{2}+2R\,\mathrm{tg}\,\vartheta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}+r_{\text{н}}-R}.$$
$$v_{\text{н}}=\sqrt{\frac{k\nu_{\text{н}}}{r_{\text{н}}}}.$$

The second diagram of calculation:

$$\beta_C=\frac{L-l_{\text{н}}}{R}.$$
$$\mathrm{tg}\,\omega=\frac{R\sin\beta_C}{r_{\text{н}}-R\cos\beta_C}.$$
$$\nu_{\text{н}}=\mathrm{tg}\frac{\beta_C}{2}\cdot\frac{\sin\omega}{\cos\vartheta_{\text{н}}\cos(\omega-\vartheta_{\text{н}})}.$$
$$v_{\text{н}}=\sqrt{\frac{k\nu_{\text{н}}}{r_{\text{н}}}}.$$

In a particular case, if $r = r_H$ and $\beta/2 = \beta_B$, then from equation (19.17) we obtain

$$v_{\text{н}} = \frac{1+\operatorname{tg}^2\vartheta_{\text{н}}}{\operatorname{tg}\beta_{\text{в}}+\operatorname{tg}\vartheta_{\text{н}}}\operatorname{tg}\beta_{\text{в}}.$$

3. To find from the assigned distance L and altitude h_H the optimum angle $\vartheta_{H\ \text{опт}}$ requiring the minimum speed $v_{H\min}$.

From equation (19.16) we have

$$v_{\text{н}\min} = \sqrt{\frac{k\nu_{\text{н}\min}}{r_{\text{н}}}}. \qquad (19.20)$$

Consequently, it is necessary to fine $\nu_{H\min}$. This can be done by two methods.

First method. Let us differentiate expression (19.9), examining ν_H as the implicit function from tg ϑ_H, and let us equate to zero $\frac{d\nu_{\text{н}}}{d\operatorname{tg}\vartheta_{\text{н}}}$. Considering $r = r_C$ and $\beta = \beta_C$, we will obtain

$$2r_C\nu_{\text{н}\min}\operatorname{tg}\frac{\beta_C}{2} - 4r_C\operatorname{tg}^3\frac{\beta_C}{2}\operatorname{tg}\vartheta_{\text{н опт}} = 0,$$

i.e.,

$$\nu_{\text{н}\min} = 2\operatorname{tg}\frac{\beta_C}{2}\operatorname{tg}\vartheta_{\text{н опт}} \qquad (19.21)$$

Solving jointly equations (19.9) and (19.21, we will find

$$\left[(r_{\text{н}}+r_C)\operatorname{tg}^2\frac{\beta_C}{2}+2r_C\operatorname{tg}\vartheta_{\text{н опт}}\operatorname{tg}\frac{\beta_C}{2}+r_{\text{н}}-r_C\right]2\operatorname{tg}\frac{\beta_C}{2}\operatorname{tg}\vartheta_{\text{н опт}} =$$
$$= 2r_C\operatorname{tg}^2\frac{\beta_C}{2}(1+\operatorname{tg}^2\vartheta_{\text{н опт}}),$$

or

$$\left[(r_{\text{н}}+r_C)\operatorname{tg}^2\frac{\beta_C}{2}+r_{\text{н}}-r_C\right]\operatorname{tg}\vartheta_{\text{н опт}} = r_C\operatorname{tg}\frac{\beta_C}{2}(1-\operatorname{tg}^2\vartheta_{\text{н опт}}),$$

what can be rewritten in the form

$$\operatorname{tg}2\vartheta_{\text{н опт}} = \frac{2\operatorname{tg}\vartheta_{\text{н опт}}}{1-\operatorname{tg}^2\vartheta_{\text{н опт}}} = \frac{2r_C\operatorname{tg}\frac{\beta_C}{2}}{(r_{\text{н}}+r_C)\operatorname{tg}^2\frac{\beta_C}{2}+r_{\text{н}}-r_C}. \qquad (19.22)$$

Calculation of the angle $\vartheta_{H\ \text{опт}}$ is considerably simplified if one were to introduce the angle

$$\varepsilon = 2\vartheta_{\text{н опт}} + \frac{\beta_C}{2}. \qquad (19.23)$$

Then we will find:

$$\operatorname{tg}\varepsilon=\frac{\operatorname{tg}2\vartheta_{\text{н опт}}+\operatorname{tg}\frac{\beta_c}{2}}{1-\operatorname{tg}2\vartheta_{\text{н опт}}\operatorname{tg}\frac{\beta_c}{2}}=\frac{\dfrac{2r_c\operatorname{tg}\frac{\beta_c}{2}}{(r_{\text{н}}+r_c)\operatorname{tg}^2\frac{\beta_c}{2}+r_{\text{н}}-r_c}+\operatorname{tg}\frac{\beta_c}{2}}{1-\dfrac{2r_c\operatorname{tg}\frac{\beta_c}{2}}{(r_{\text{н}}+r_c)\operatorname{tg}^2\frac{\beta_c}{2}+r_{\text{н}}-r_c}\operatorname{tg}\frac{\beta_c}{2}}=$$

$$=\frac{\left[(r_{\text{н}}+r_c)\operatorname{tg}^2\frac{\beta_c}{2}+r_{\text{н}}-r_c+2r_c\right]\operatorname{tg}\frac{\beta_c}{2}}{(r_{\text{н}}+r_c)\operatorname{tg}^2\frac{\beta_c}{2}+r_{\text{н}}-r_c-2r_c\operatorname{tg}^2\frac{\beta_c}{2}}=$$

$$=\frac{(r_{\text{н}}+r_c)\left(1+\operatorname{tg}^2\frac{\beta_c}{2}\right)}{(r_{\text{н}}-r_c)\left(1+\operatorname{tg}^2\frac{\beta_c}{2}\right)}\operatorname{tg}\frac{\beta_c}{2}.$$

Finally:

$$\operatorname{tg}\varepsilon=\frac{r_{\text{н}}+r_c}{r_{\text{н}}-r_c}\operatorname{tg}\frac{\beta_c}{2} \tag{19.24}$$

and from equation (19.23) we have

$$\vartheta_{\text{н опт}}=\frac{1}{2}\left(\varepsilon-\frac{\beta_c}{2}\right). \tag{19.25}$$

Knowing $\vartheta_{H\ \text{ОПТ}}$, from formulas (19.21) and (19.20) we find $\nu_{H\,\min}$ and $v_{H\,\min}$.

Second method. From equation (19.19) we obtain;

$$\nu_{\text{н}}=2\operatorname{tg}\frac{\beta_c}{2}\,\frac{\sin\omega}{\cos\omega+\cos(2\vartheta_{\text{н}}-\omega)}. \tag{19.26}$$

As can be seen from formula (19.18), the value of angle ω does not depend on ϑ_H, and therefore ν_H will be least when $\cos(2\vartheta_H-\omega)=1$, i.e., under the condition

$$2\vartheta_{\text{н опт}}=\omega. \tag{19.27}$$

Using equation (19.18), we obtain

$$\operatorname{tg}2\vartheta_{\text{н опт}}=\frac{r_c\sin\beta_c}{r_{\text{н}}-r_c\cos\beta_c}. \tag{19.28}$$

When $\cos(2\vartheta_H-\omega)=1$, from equation (19.26) we have

$$\nu_{\text{н}\,\min}=2\operatorname{tg}\frac{\beta_c}{2}\,\frac{\sin\omega}{\cos\omega+1}=2\operatorname{tg}\frac{\beta_c}{2}\operatorname{tg}\frac{\omega}{2}.$$

Considering relation (19.27), we will finally have

$$\nu_{\text{н}\,\min}=2\operatorname{tg}\frac{\beta_c}{2}\operatorname{tg}\vartheta_{\text{н опт}} \tag{19.29}$$

and by formula (19.20) we find $v_{H\,\min}$.

Thus we have the following two diagram of calculation $\vartheta_{H\ \text{опт}}$ and $\nu_{H\ \min}$ (when $r_C = R$.)

The first diagram of calculation:

$$\beta_C = \frac{L - l_{\text{н}}}{R}.$$

$$\text{tg}\,\varepsilon = \frac{r_{\text{н}} + R}{r_{\text{н}} - R}\,\text{tg}\,\frac{\beta_C}{2}.$$

$$\vartheta_{\text{н опт}} = \frac{1}{2}\left(\varepsilon - \frac{\beta_C}{2}\right).$$

$$\nu_{\text{н}\,\min} = 2\,\text{tg}\,\frac{\beta_C}{2}\,\text{tg}\,\vartheta_{\text{н опт}}.$$

$$v_{\text{н}\,\min} = \sqrt{\frac{k\nu_{\text{н}\,\min}}{r_{\text{н}}}}.$$

The second diagram of calculation:

$$\beta_C = \frac{L - l_{\text{н}}}{R}.$$

$$\text{tg}\,2\vartheta_{\text{н опт}} = \frac{R\sin\beta_C}{r_{\text{н}} - R\cos\beta_C}.$$

$$\nu_{\text{н}\,\min} = 2\,\text{tg}\,\frac{\beta_C}{2}\,\text{tg}\,\vartheta_{\text{н опт}}.$$

$$v_{\text{н}\,\min} = \sqrt{\frac{k\nu_{\text{н}\,\min}}{r_{\text{н}}}}.$$

In the particular case when $r_C = r_H$ and $\frac{\beta}{2} = \beta_B$, from formula (19.24) we have $\text{tg}\,\varepsilon = \infty$, i.e., $\varepsilon = 90^\circ$. Consequently, from equation (19.25) we obtain

$$\vartheta_{\text{н опт}} = 45^\circ - \frac{\beta_C}{4} = 45^\circ - \frac{\beta_{\text{в}}}{2}. \tag{19.30}$$

4. To find from the assigned speed v_H and altitude h_H the optimum angle $\vartheta_{H\ \text{опт}}$, providing the maximum range $L_{\max}$.

Differentiating expression (19.9), considering $\text{tg}\,\beta$ of the implicit function from $\text{tg}\,\vartheta_H$ and equating the derivative $\frac{d\,\text{tg}\,\beta}{d\,\text{tg}\,\vartheta_{\text{н}}}$ to zero, we will obtain (when $r = r_C$ and $\beta = \beta_C$)

$$2r_C\,\text{tg}^2\frac{\beta_{C\max}}{2}\cdot 2\,\text{tg}\,\vartheta_{\text{н опт}} - 2\nu_{\text{н}} r_C\,\text{tg}\,\frac{\beta_{C\max}}{2} = 0,$$

i.e.,

$$\text{tg}\,\frac{\beta_{C\max}}{2} = \frac{\nu_{\text{н}}}{2\,\text{tg}\,\vartheta_{\text{н опт}}}. \tag{19.31}$$

Solving jointly the equations (19.9) and (19.31), we will find

$$[2r_C(1 + \text{tg}^2\vartheta_{\text{н опт}}) - (r_{\text{н}} + r_C)\nu_{\text{н}}]\cdot\frac{\nu_{\text{н}}^2}{4\,\text{tg}^2\vartheta_{\text{н опт}}} - \\ - 2\nu_{\text{н}} r_C\,\text{tg}\,\vartheta_{\text{н опт}}\frac{\nu_{\text{н}}}{2\,\text{tg}\,\vartheta_{\text{н опт}}} - (r_{\text{н}} - r_C)\nu_{\text{н}} = 0.$$

or

$$2r_C\nu_{\text{н}} + 2r_C\nu_{\text{н}}\,\text{tg}^2\vartheta_{\text{н опт}} - (r_{\text{н}} + r_C)\nu_{\text{н}}^2 - \\ - 4\nu_{\text{н}} r_C\,\text{tg}^2\vartheta_{\text{н опт}} - (r_{\text{н}} - r_C)\cdot 4\,\text{tg}^2\vartheta_{\text{н опт}} = 0.$$

i.e.,

$$[2\nu_{\text{н}} r_C + 4(r_{\text{н}} - r_C)]\,\text{tg}^2\,\theta_{\text{н опт}} = \nu_{\text{н}}[2r_C - (r_{\text{н}} + r_C)\nu_{\text{н}}].$$

Consequently,

$$\text{tg}^2\,\theta_{\text{н опт}} = \frac{\nu_{\text{н}}[2r_C - (r_{\text{н}} + r_C)\nu_{\text{н}}]}{2[\nu_{\text{н}} r_C + 2(r_{\text{н}} - r_C)]}. \tag{19.32}$$

Thus the given problem can be solved by the following diagram (when $r_C = R$):

$$\nu_{\text{н}} = \frac{v_{\text{н}}^2 r_{\text{н}}}{k},$$

$$\text{tg}\,\theta_{\text{н опт}} = \sqrt{\frac{\nu_{\text{н}}}{2}\cdot\frac{2R - (r_{\text{н}} + R)\nu_{\text{н}}}{\nu_{\text{н}} R + 2(r_{\text{н}} - R)}},$$

$$\text{tg}\,\frac{\beta_{C\,\max}}{2} = \frac{\nu_{\text{н}}}{2\,\text{tg}\,\theta_{\text{н опт}}},$$

$$l_{\text{эл}\,\max} = R\beta_{C\,\max},$$

$$L_{\max} = l_{\text{н}} + l_{\text{эл}\,\max}.$$

For the particular case $r_H = r_C$ from equation (19.32) we have

$$\text{tg}\,\theta_{\text{н опт}} = \sqrt{1 - \nu_{\text{н}}}. \tag{19.33}$$

5. Determining the change in distance L depending upon small changes of altitude h_H, speed v_H and angle ϑ_H.

Obviously, it is necessary to find derivatives

$$\frac{\partial\beta_C}{\partial r_{\text{н}}},\quad \frac{\partial\beta_C}{\partial v_{\text{н}}}\quad\text{and}\quad \frac{\partial\beta_C}{\partial\theta_{\text{н}}}.$$

We will use equation (19.9), and namely,

$$[2r_C(1 + \text{tg}^2\,\theta_{\text{н}}) - (r_{\text{н}} + r_C)\nu_{\text{н}}]\,\text{tg}^2\,\frac{\beta_C}{2} - 2\nu_{\text{н}} r_C\,\text{tg}\,\theta_{\text{н}}\,\text{tg}\,\frac{\beta_C}{2} - (r_{\text{н}} - r_C)\nu_{\text{н}} = 0.$$

In common form this equation will be thus recorded:

$$F(r_{\text{н}}, v_{\text{н}}, \theta_{\text{н}}, \beta_C) = 0.$$

Let us differentiate:

$$\frac{\partial F}{\partial r_{\text{н}}}dr_{\text{н}} + \frac{\partial F}{\partial v_{\text{н}}}dv_{\text{н}} + \frac{\partial F}{\partial \theta_{\text{н}}}d\theta_{\text{н}} + \frac{\partial F}{\partial \beta_C}d\beta_C = 0. \tag{19.34}$$

Inasmuch as ν_H depends on v_H and r_H:

$$\nu_\text{н} = f(v_\text{н},\ r_\text{н}),$$

then

$$d\nu_\text{н} = \frac{\partial f}{\partial r_\text{н}}\,dr_\text{н} + \frac{\partial f}{\partial v_\text{н}}\,dv_\text{н}. \tag{19.35}$$

Substituting (19.35) into (19.34), we obtain

$$\left(\frac{\partial F}{\partial r_\text{н}} + \frac{\partial F}{\partial \nu_\text{н}}\frac{\partial f}{\partial r_\text{н}}\right)dr_\text{н} + \frac{\partial F}{\partial \nu_\text{н}}\frac{\partial f}{\partial v_\text{н}}\,dv_\text{н} + \frac{\partial F}{\partial \theta_\text{н}}\,d\theta_\text{н} + \frac{\partial F}{\partial \beta_C}\,d\beta_C = 0.$$

Hence

$$d\beta_C = -\frac{1}{\dfrac{\partial F}{\partial \beta_C}}\left(\frac{\partial F}{\partial r_\text{н}} + \frac{\partial F}{\partial \nu_\text{н}}\frac{\partial f}{\partial r_\text{н}}\right)dr_\text{н} - \frac{\dfrac{\partial F}{\partial \nu_\text{н}}\dfrac{\partial f}{\partial v_\text{н}}}{\dfrac{\partial F}{\partial \beta_C}}\,dv_\text{н} - \frac{\dfrac{\partial F}{\partial \theta_\text{н}}}{\dfrac{\partial F}{\partial \beta_C}}\,d\theta_\text{н}$$

and, consequently,

$$\frac{\partial \beta_C}{\partial r_\text{н}} = -\left(\frac{\partial F}{\partial r_\text{н}} + \frac{\partial F}{\partial \nu_\text{н}}\frac{\partial f}{\partial r_\text{н}}\right)\Big/\frac{\partial F}{\partial \beta_C}, \tag{19.36}$$

$$\frac{\partial \beta_C}{\partial v_\text{н}} = -\frac{\partial F}{\partial \nu_\text{н}}\frac{\partial f}{\partial v_\text{н}}\Big/\frac{\partial F}{\partial \beta_C}, \tag{19.37}$$

$$\frac{\partial \beta_C}{\partial \theta_\text{н}} = -\frac{\partial F}{\partial \theta_\text{н}}\Big/\frac{\partial F}{\partial \beta_C}. \tag{19.38}$$

We find derivatives

$$\frac{\partial F}{\partial r_\text{н}} = -\nu_\text{н}\operatorname{tg}^2\frac{\beta_C}{2} - \nu_\text{н} = -\frac{\nu_\text{н}}{\cos^2\dfrac{\beta_C}{2}},$$

$$\frac{\partial F}{\partial \nu_\text{н}} = -(r_\text{н} + r_C)\operatorname{tg}^2\frac{\beta_C}{2} - 2r_C\operatorname{tg}\theta_\text{н}\operatorname{tg}\frac{\beta_C}{2} - (r_\text{н} - r_C).$$

But from equation (19.17) we have

$$(r_\text{н} + r_C)\operatorname{tg}^2\frac{\beta_C}{2} + 2r_C\operatorname{tg}\theta_\text{н}\operatorname{tg}\frac{\beta_C}{2} + r_\text{н} - r_C = \frac{2r_C(1 + \operatorname{tg}^2\theta_\text{н})\operatorname{tg}^2\dfrac{\beta_C}{2}}{\nu_\text{н}}.$$

Consequently,

$$\frac{\partial F}{\partial \nu_\text{н}} = -\frac{2r_C}{\nu_\text{н}}(1 + \operatorname{tg}^2\theta_\text{н})\operatorname{tg}^2\frac{\beta_C}{2}.$$

Since $\nu_к = \frac{v_к^2 r_к}{k}$, then

$$\frac{\partial f}{\partial r_к} = \frac{v_к^2}{k} = \frac{\nu_к}{r_к}$$

and

$$\frac{\partial f}{\partial v_к} = \frac{2v_к r_к}{k} = \frac{2\nu_к}{v_к}.$$

Further,

$$\frac{\partial F}{\partial \beta_C} = \frac{[2r_C(1+\operatorname{tg}^2\theta_к)-(r_к+r_C)\nu_к]\operatorname{tg}\frac{\beta_C}{2}-\nu_к r_C\operatorname{tg}\theta_к}{\cos^2\frac{\beta_C}{2}}.$$

But from equation (19.9) we have

$$[2r_C(1+\operatorname{tg}^2\theta_к)-(r_к+r_C)\nu_к]\operatorname{tg}\frac{\beta_C}{2}-\nu_к r_C\operatorname{tg}\theta_к = \\ = \nu_к\left(r_к-r_C+r_C\operatorname{tg}\theta_к\operatorname{tg}\frac{\beta_C}{2}\right)\frac{1}{\operatorname{tg}\frac{\beta_C}{2}}.$$

Consequently,

$$\frac{\partial F}{\partial \beta_C} = \frac{\nu_к\left(r_к-r_C+r_C\operatorname{tg}\theta_к\operatorname{tg}\frac{\beta_C}{2}\right)}{\operatorname{tg}\frac{\beta_C}{2}\cos^2\frac{\beta_C}{2}}$$

and, finally,

$$\frac{\partial F}{\partial \theta_к} = \frac{2r_C\operatorname{tg}\frac{\beta_C}{2}}{\cos^2\theta_к}\left(2\operatorname{tg}\frac{\beta_C}{2}\operatorname{tg}\theta_к-\nu_к\right).$$

Inserting the found derivative into espressions (19.36)-(19.38), we will obtain:

$$\frac{\partial \beta_C}{\partial r_к} = \frac{\operatorname{tg}\frac{\beta_C}{2}\cos^2\frac{\beta_C}{2}}{\nu_к\left(r_к-r_C+r_C\operatorname{tg}\theta_к\operatorname{tg}\frac{\beta_C}{2}\right)}\times \\ \times\left[\frac{\nu_к}{\cos^2\frac{\beta_C}{2}}+2\frac{r_C}{r_к}\operatorname{tg}^2\frac{\beta_C}{2}(1+\operatorname{tg}^2\theta_к)\right] = \\ = \frac{\nu_к+\frac{2r_C}{r_к}(1+\operatorname{tg}^2\theta_к)\sin^2\frac{\beta_C}{2}}{\nu_к\left(r_к-r_C+r_C\operatorname{tg}\theta_к\operatorname{tg}\frac{\beta_C}{2}\right)}\operatorname{tg}\frac{\beta_C}{2}. \qquad (19.39)$$

$$\frac{\partial\beta_C}{\partial v_{\text{н}}}=\frac{2r_C\,\mathrm{tg}^2\frac{\beta_C}{2}\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\mathrm{tg}\frac{\beta_C}{2}\cos^2\frac{\beta_C}{2}\,2v_{\text{н}}}{v_{\text{н}}^2\left(r_{\text{н}}-r_C+r_C\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)v_{\text{н}}}=$$

$$=\frac{4r_C}{v_{\text{н}}}\cdot\frac{\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\sin^2\frac{\beta_C}{2}\,\mathrm{tg}\frac{\beta_C}{2}}{v_{\text{н}}\left(r_{\text{н}}-r_C+r_C\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)}. \qquad (19.40)$$

$$\frac{\partial\beta_C}{\partial\theta_{\text{н}}}=\frac{2r_C\,\mathrm{tg}\frac{\beta_C}{2}\left(v_{\text{н}}-2\,\mathrm{tg}\frac{\beta_C}{2}\,\mathrm{tg}\,\theta_{\text{н}}\right)\mathrm{tg}\frac{\beta_C}{2}\cos^2\frac{\beta_C}{2}}{\cos^2\theta_{\text{н}}\cdot v_{\text{н}}\left(r_{\text{н}}-r_C+r_C\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)}=$$

$$=\frac{2r_C\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\left(v_{\text{н}}-2\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)\sin^2\frac{\beta_C}{2}}{v_{\text{н}}\left(r_{\text{н}}-r_C+r_C\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)}. \qquad (19.41)$$

Thus for the solution of the given problem we have the following formulas (when $r_C = R$):

$$\left.\begin{aligned}
\frac{\partial l_{\text{сп}}}{\partial r_{\text{н}}}&=R\,\frac{v_{\text{н}}+\frac{2R}{r_{\text{н}}}\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\sin^2\frac{\beta_C}{2}\,\mathrm{tg}\frac{\beta_C}{2}}{v_{\text{н}}\left(r_{\text{н}}-R+R\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)},\\
\frac{\partial l_{\text{сп}}}{\partial v_{\text{н}}}&=\frac{4R^2}{v_{\text{н}}}\cdot\frac{\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\sin^2\frac{\beta_C}{2}\,\mathrm{tg}\frac{\beta_C}{2}}{v_{\text{н}}\left(r_{\text{н}}-R+R\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)},\\
\frac{\partial l_{\text{сп}}}{\partial\theta_{\text{н}}}&=2R^2\,\frac{\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\left(v_{\text{н}}-2\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)\sin^2\frac{\beta_C}{2}}{v_{\text{н}}\left(r_{\text{н}}-R+R\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)}.
\end{aligned}\right\} \qquad (19.42)$$

In the particular case $r_C = r_H$ from equations (19.39)-(1941) we easily obtain the following formulas:

$$\frac{\partial\beta_C}{\partial r_{\text{н}}}=\frac{v_{\text{н}}+2\sin^2\frac{\beta_C}{2}\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)}{v_{\text{н}}r_{\text{н}}\,\mathrm{tg}\,\theta_{\text{н}}},$$

$$\frac{\partial\beta_C}{\partial v_{\text{н}}}=\frac{4\sin^2\frac{\beta_C}{2}\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)}{v_{\text{н}}v_{\text{н}}\,\mathrm{tg}\,\theta_{\text{н}}},$$

$$\frac{\partial\beta_C}{\partial\theta_{\text{н}}}=\frac{\left(v_{\text{н}}-2\,\mathrm{tg}\,\theta_{\text{н}}\,\mathrm{tg}\frac{\beta_C}{2}\right)\left(1+\mathrm{tg}^2\theta_{\text{н}}\right)\sin\beta_C}{v_{\text{н}}\,\mathrm{tg}\,\theta_{\text{н}}}.$$

6. Determining parameters of motion h, v, ϑ, t at any point of the trajectory determined by the central angle β.

a) Altitude of flight above the surface of earth at any point of the trajectory is equal to

$$h = r - R.$$

For the determination of r we have expression (18.25):

$$r = \frac{p}{1 - e\cos(\beta_{\text{в}} - \beta)};$$

with this we find p and e by the formulas (19.3) and (18.29), namely,

$$p = \nu_{\text{н}} r_{\text{н}} \cos^2\vartheta_{\text{н}},$$

$$e = \sqrt{1 - (2 - \nu_{\text{н}})\nu_{\text{н}}\cos^2\vartheta_{\text{н}}}.$$

The value of β_B will be found from equations (19.6) and (19.7):

$$\operatorname{tg}\beta_{\text{в}} = \frac{\nu_{\text{н}}\sin\vartheta_{\text{н}}\cos\vartheta_{\text{н}}}{1 - \nu_{\text{н}}\cos^2\vartheta_{\text{н}}} = \frac{\nu_{\text{н}}\operatorname{tg}\vartheta_{\text{н}}}{1 + \operatorname{tg}^2\vartheta_{\text{н}} - \nu_{\text{н}}}.$$

The following formulas are also useful. From (18.11) we obtain

$$c_2 = \frac{v^2 r - 2k}{r},$$

whence, taking into account designation (18.26),

$$\frac{c_2}{k} = \frac{\nu - 2}{r}.$$

On the other hand, from (18.19) and (18.20) it follows:

$$\frac{c_2}{k} = \frac{e^2 - 1}{p}.$$

Hence we obtain:

$$\frac{r}{2 - \nu} = \frac{p}{1 - e^2}, \qquad (19.43)$$

in particular,

$$\frac{r_{\text{н}}}{2 - \nu_{\text{н}}} = \frac{p}{1 - e^2}. \qquad (19.44)$$

Formula (19.43) permits easily expressing any of the quantities r, ν, p, e in terms of the three others.

From (18.25) we obtain

$$1 - e\cos(\beta_{\text{в}} - \beta) = \frac{p}{r}$$

and, on the basis of (19.43),

$$1 - e\cos(\beta_{\text{в}} - \beta) = \frac{1 - e^2}{2 - \nu}. \qquad (19.45)$$

whence

$$\cos(\beta_{\text{в}} - \beta) = \frac{1 + e^2 - \nu}{e(2 - \nu)},$$

$$\nu = \frac{1 - 2e\cos(\beta_{\text{в}} - \beta) + e^2}{1 - e\cos(\beta_{\text{в}} - \beta)}. \qquad (19.46).$$

b) Speed of flight at any point of the trajectory can be found if one uses formula (18.26):

$$v = \sqrt{\frac{k\nu}{r}}.$$

If one were to express ν in terms of r with the help of formula (19.43), then we will obtain

$$v = \sqrt{k\left(\frac{2}{r} - \frac{1-e^2}{p}\right)}.$$

which it is possible with the help of formulas (19.44) and (18.28) to write in the form

$$v = \sqrt{v_a^2 + 2k\left(\frac{1}{r} - \frac{1}{r_a}\right)}. \qquad (19.47)$$

It is possible also on the basis of formulas (18.25) and (19.46) to obtain for speed v the expression directly in terms of angle β:

$$v = \sqrt{\frac{k}{p}\left(1 - 2e\cos(\beta_a - \beta) + e^2\right)}.$$

c) We find the angle of inclination of the velocity vector to the local horizon ϑ, noticing that on the basis of formulas (19.5) and (19.4)

$$\nu\sin\vartheta\cos\vartheta = e\sin(\beta_a - \beta),$$
$$\nu\cos^2\vartheta = 1 - e\cos(\beta_a - \beta).$$

whence

$$\operatorname{tg}\vartheta = \frac{e\sin(\beta_a - \beta)}{1 - e\cos(\beta_a - \beta)}. \qquad (19.48)$$

From relation (19.2) we obtain another simple formula:

$$\cos\vartheta = \sqrt{\frac{p}{w}}. \qquad (19.49)$$

d) Time of flight to any point of the trajectory is determined from equation (18.3):

$$r^2\frac{d\chi}{dt} = c_1.$$

Dividing the variables and integrating, we will obtain

$$t = \frac{1}{c_1}\int_{\chi_a}^{\chi} r^2\,d\chi.$$

From equation (18.22)

$$\chi = \chi_a + \beta$$
$$d\chi = d\beta.$$

Considering also (18.25), we will have

$$t = \frac{p^2}{c_1}\int_0^{\beta}\frac{d\beta}{[1 - e\cos(\beta_a - \beta)]^2}. \qquad (19.50)$$

We accomplish integration in the following way. Let us introduce the new variable x connected with β by the relation

$$\cos(\beta_{в}-\beta)=\frac{e+\cos x}{1+e\cos x}. \tag{19.51}$$

Differentiating, we obtain

$$\sin(\beta_{в}-\beta)\,d\beta=\frac{(1+e\cos x)(-\sin x)-(e+\cos x)(-e\sin x)}{(1+e\cos x)^2}\,dx=$$
$$=-\frac{(1-e^2)\sin x}{(1+e\cos x)^2}\,dx. \tag{19.52}$$

For the elimination of sin (β_B - β) let us use relation (19.51):

$$\sin^2(\beta_{в}-\beta)=1-\frac{e^2+2e\cos x+\cos^2 x}{1+2e\cos x+e^2\cos^2 x}=\frac{(1-e^2)(1-\cos^2 x)}{(1+e\cos x)^2},$$

whence

$$\sin(\beta_{в}-\beta)=\pm\frac{\sqrt{1-e^2}\sin x}{1+e\cos x}. \tag{19.53}$$

So that the connection between β and x is single-valued, let us dwell in formula (19.53) on one of the signs, namely, on the minus sign; then from relations (19.52) and (19.53) we will obtain

$$d\beta=\frac{\sqrt{1-e^2}}{1+e\cos x}\,dx. \tag{19.54}$$

Further from expression (19.51) we find

$$1-e\cos(\beta_{в}-\beta)=\frac{1-e^2}{1+e\cos x}, \tag{19.55}$$

which permits recording the integral (19.50) in the form

$$t=\frac{p^2}{c_1}\int_{x_{в}}^{x}\frac{(1+e\cos x)^2}{(1-e^2)^2}\,\frac{\sqrt{1-e^2}}{1+e\cos x}\,dx=$$
$$=\frac{p^2}{c_1(1-e^2)^{3/2}}\int_{x_{в}}^{x}(1+e\cos x)\,dx=$$
$$=\frac{p^2}{c_1(1-e^2)^{3/2}}(x+e\sin x)\Big|_{x_{в}}^{x}. \tag{19.56}$$

Relation (19.53) (taking into account the selected sign) and (19.55) give

$$\sin x=-\frac{\sqrt{1-e^2}\sin(\beta_{в}-\beta)}{1-e\cos(\beta_{в}-\beta)}.$$

Comparing this formula with formula (19.48), we obtain

$$\sin x=-\frac{\sqrt{1-e^2}}{e}\operatorname{tg}\vartheta. \tag{19.57}$$

From a comparison of formulas (19.45) and (19.55) we also find

$$2-\nu = 1 + e\cos x,$$

whence

$$\cos x = \frac{1-\nu}{e}.$$

On the basis of the obtained formulas it is possible to write:

$$x = \begin{cases} -\arccos\dfrac{1-\nu}{e} & \text{on the ascending phase of trajectory,} \\ \arccos\dfrac{1-\nu}{e} & \text{on descending phase.} \end{cases} \qquad (19.58)$$

Let us substitute expression (19.57) for sin x into formula (19.56):

$$t = \frac{p^2}{\varepsilon_1 (1-e^2)^{3/2}}\left(x - \sqrt{1-e^2}\,\mathrm{tg}\,\theta - x_{\text{н}} + \sqrt{1-e^2}\,\mathrm{tg}\,\theta_{\text{н}}\right) = \\ = \frac{p^2}{\varepsilon_1 (1-e^2)}\left(\frac{x-x_{\text{н}}}{\sqrt{1-e^2}} - \mathrm{tg}\,\theta + \mathrm{tg}\,\theta_{\text{н}}\right).$$

A constant coefficient in this formula it is possible by using formula (18.19) and also to transform (19.44) and (19.2) to the form

$$\frac{p^2}{\varepsilon_1 (1-e^2)} = \frac{p}{1-e^2}\sqrt{\frac{p}{k}} = \frac{r_{\text{н}} v_{\text{н}} \cos\theta_{\text{н}}}{\nu_{\text{н}}(2-\nu_{\text{н}})},$$

so that we finally obtain:

$$t = \frac{p}{1-e^2}\sqrt{\frac{p}{k}}\left(\frac{x-x_{\text{н}}}{\sqrt{1-e^2}} + \mathrm{tg}\,\theta_{\text{н}} - \mathrm{tg}\,\theta\right) = \\ = \frac{r_{\text{н}} v_{\text{н}}}{\nu_{\text{н}}(2-\nu_{\text{н}})}\left(\frac{x-x_{\text{н}}}{\sqrt{(2-\nu_{\text{н}})\nu_{\text{н}}}} + \sin\theta_{\text{н}} - \cos\theta_{\text{н}}\,\mathrm{tg}\,\theta\right). \qquad (19.59)$$

where x and x_H are determined by the formula (19.58). For the most widespread case when the initial point of the section of free flight is located on the ascending and the final point on the descending phase of the trajectory, we obtain

$$t = \frac{p}{1-e^2}\sqrt{\frac{p}{k}}\left(\frac{\arccos\dfrac{1-\nu}{e} + \arccos\dfrac{1-\nu_{\text{н}}}{e}}{\sqrt{1-e^2}} + \mathrm{tg}\,\theta_{\text{н}} - \mathrm{tg}\,\theta\right). \qquad (19.60)$$

Thus we have the following diagram of calculation for the determination of parameters of the trajectory at any point of it assigned by the central angle β:

1) $\nu_{\text{н}} = \frac{v_{\text{н}}^2 r_{\text{н}}}{k}$;

2) $\mathrm{tg}\,\beta_{\text{н}} = \frac{\nu_{\text{н}}\,\mathrm{tg}\,\theta_{\text{н}}}{1+\mathrm{tg}^2\theta_{\text{н}} - \nu_{\text{н}}}$,

$0 < \beta_{\text{н}} < \pi$ when $\theta_{\text{н}} > 0$, $-\pi < \beta_{\text{н}} < 0$ when $\theta_{\text{н}} < 0$;

3) $p = r_{\text{н}}\nu_{\text{н}}\cos^2\theta_{\text{н}}$;

4) $e = \sqrt{1-(2-\nu_{\text{н}})\nu_{\text{н}}\cos^2\theta_{\text{н}}}$;

5) $\cos x_H = \frac{1-\nu_H}{e}$, sign of sin x_H is opposite to sign of sin ϑ_H;

6) $r = \frac{p}{1 - e\cos(\beta_B - \beta)}$;

7) $h = r - R$;

8) $\nu = 2 - \frac{r(1-e^2)}{p}$;

9) $v = \sqrt{\frac{k\nu}{r}}$;

10) $\cos\vartheta = \sqrt{\frac{p}{r\nu}}$, sign of sin ϑ coincides with the sign of sin $(\beta_B - \beta)$;

11) $\cos x = \frac{1-\nu}{e}$, sign of sin x is opposite to sign of sin $(\beta_B - \beta)$;

12) $t = \frac{p}{1-e^2}\sqrt{\frac{p}{k}}\left(\frac{x - x_H}{\sqrt{1-e^2}} + \operatorname{tg}\vartheta_H - \operatorname{tg}\vartheta\right)$.

Particular Case 1. Parameters of motion at the peak of the trajectory. Altitude at the peak is determined thus:

$$h_B = r_B - R.$$

(subscript "B" corresponds to peak of trajectory).

When $\beta = \beta_B$ from equation (18.25) we obtain

$$r_B = \frac{p}{1-e}.$$

Considering (19.44), we will find

$$r_B = \frac{r_H}{2-\nu_H}(1+e). \quad (19.61)$$

The angle of inclination of the tangent in the peak of the trajectory is equal to zero.

To determine the speed at the peak from equations (18.4) and (18.5) we have

$$r_H v_H \cos\vartheta_H = r_B v_B \cos\vartheta_B;$$

since $\vartheta_B = 0$, then

$$v_B = \frac{r_H v_H \cos\vartheta_H}{r_B}.$$

Considering equation (19.61), we will obtain

$$v_B = \frac{(2-\nu_H)\, v_H \cos\vartheta_H}{1+e}. \quad (19.62)$$

Further from expression (19.46) it follows

$$\nu_B = 1 - e,$$

and from formulas (19.58) we obtain

$$x_B = 0.$$

Inserting this value into formula (19.59) and considering that $\vartheta_B = 0$, we will obtain the following expression for the time of flight up to the peak of the trajectory:

$$t_{\text{в}} = \frac{p}{1-e^2}\sqrt{\frac{p}{k}}\left(\operatorname{tg}\theta_{\text{в}} - \frac{e_{\text{в}}}{\sqrt{1-e^2}}\right) =$$

$$= \frac{r_{\text{н}}v_{\text{н}}}{v_{\text{н}}(2-\nu_{\text{н}})}\left(\sin\theta_{\text{в}} - \frac{e_{\text{в}}}{\sqrt{(2-\nu_{\text{н}})\nu_{\text{н}}}}\right). \qquad (19.63)$$

Particular Case 2. Vertical launching. For this case we find the maximum altitude and time of flight up to the peak. Since with vertical launching $\vartheta_H = \pi/2$, then

$$e = \sqrt{1-(2-\nu_{\text{н}})\nu_{\text{н}}\cdot 0} = 1.$$

Consequently, from equation (19.61) we obtain

$$\left.\begin{aligned} r_{\text{в.п}} &= \frac{2r_{\text{н}}}{2-\nu_{\text{н}}}, \\ h_{\text{в.п}} &= r_{\text{в.п}} - R, \end{aligned}\right\} \qquad (19.64)$$

where subscript "в. п" corresponds to the case of vertical launching.

We find the time of flight up to the peak from equation (19.63), considering $\vartheta_H = \pi/2$ and considering formula (19.58):

$$t_{\text{в.п}} = \frac{r_{\text{н}}\nu_{\text{н}}}{v_{\text{н}}(2-\nu_{\text{н}})}\left(1 + \frac{\arccos(1-\nu_{\text{н}})}{\sqrt{(2-\nu_{\text{н}})\nu_{\text{н}}}}\right). \qquad (19.65)$$

or, in another form,

$$t_{\text{в.п}} = \frac{r_{\text{н}}}{v_{\text{н}}(2-\nu_{\text{н}})}\left(\nu_{\text{н}} + \sqrt{\frac{\nu_{\text{н}}}{2-\nu_{\text{н}}}}\arccos(1-\nu_{\text{н}})\right).$$

Fig. 19.1, 19.2, and 19.3 depict the families of characteristic flight paths of the rocket. Trajectories on Fig. 19.1 are obtained at a constant initial speed but at different angles of departure ϑ_H. Fig. 19.2 shows the trajectories obtained at an optimum angle of departure for different $\nu_H < 1$. Fig. 19.3 depicts trajectories at $\nu_H > 1$ and $\vartheta_H = 0$.

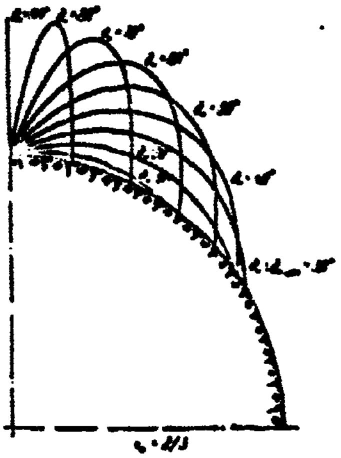

Fig. 19.1

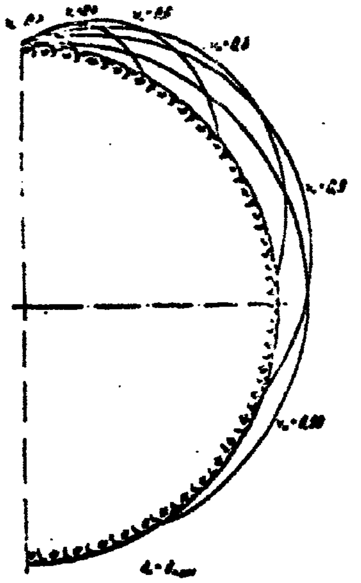

Fig. 19.2

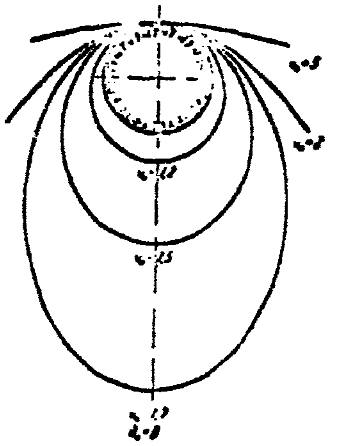

Fig. 19.3

§ 20. Return to Relative Motion

The formulas obtained above permit solving the problem about the absolute motion of the rocket. In examining the relative motion these formulas are useful only for the determination of the mean trajectory, from which deviations to various sides are possible due to the rotation of earth. But in many cases absolute motion is interesting not by itself, but by the fact that from it it is possible to turn to the relative motion, considering, along with the motion of the rocket, the rotation of earth.

By knowing the geographic coordinates of the initial point of free flight φ_{TH} and λ_H, absolute azimuth ψ_H and the central angle passed by the rocket in absolute motion β', it is easy to find the geographic coordinates φ'_r and λ' of point P, on the assumption that earth is motionless, by the formulas of sines and cosines of spheric trigonometry [1] (Fig. 20.1):

$$\cos(90^\circ - \varphi'_г) = $$
$$= \cos(90^\circ - \varphi_{гн})\cos\beta' + \sin(90^\circ - \varphi_{гн})\sin\beta'\cos\psi'_н,$$

or

$$\sin\varphi'_г = \sin\varphi_{гн}\cos\beta' + \cos\varphi_{гн}\sin\beta'\cos\psi'_н, \tag{20.1}$$

and

$$\frac{\sin(\lambda' - \lambda_н)}{\sin\beta'} = \frac{\sin\psi'_н}{\cos\varphi'_г},$$

whence

$$\sin(\lambda' - \lambda_н) = \frac{\sin\beta'\sin\psi'_н}{\cos\varphi'_г}. \tag{20.2}$$

---

[1] The basic formulas of spheric trigonometry are derived in the following way. Let us consider on a sphere unit radius with the center at point O the triangle ΔABC, formed by arcs of great circles (Fig. 20.2). Let us construct an auxiliary system of coordinates Oxyz, directing the axis Ox about the radius OA and combining plane Oxy with the plane OAB. Radii vectors of points A, B, C will have in the system Oxyz these components:

$$\overline{OA}(1, 0, 0),$$
$$\overline{OB}(\cos c, \sin c, 0),$$
$$\overline{OC}(\cos b, \sin b\cos A, \sin b\sin A):$$

Calculating the scalar product of unit vectors $\overline{OB}$ and $\overline{OC}$, we will obtain

$$\cos a = \cos b\cos c + \sin b\sin c\cos A.$$

This formula is called the formula of cosines.

Let us calculate the mixed product of the three vectors $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$:

$$\overline{OA}\cdot(\overline{OB}\times\overline{OC}) = \begin{vmatrix} 1 & 0 & 0 \\ \cos c & \sin c & 0 \\ \cos b & \sin b\cos A & \sin b\sin A \end{vmatrix} = \sin b\sin c\sin A.$$

From considerations of symmetry it is possible to write two other expressions:

$$\overline{OA}\cdot(\overline{OB}\times\overline{OC}) = \sin a\sin c\sin B = \sin a\sin b\sin C.$$

Dividing by (sin a sin b sin c), we obtain the formula of sines

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

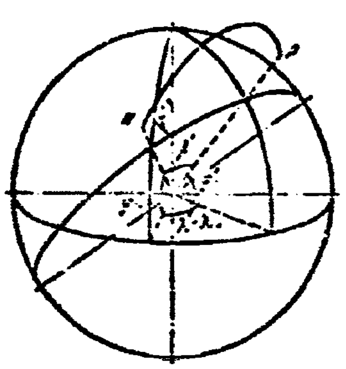

Fig. 20.1

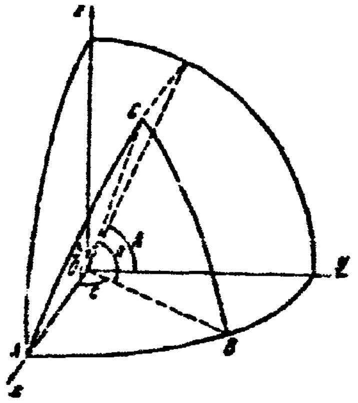

Fig. 20.2

We consider now that earth revolves simultaneously with the motion of the rocket. As a result of this rotation the point with the geographic coordinates φ_r' and λ_r' will depart in parallel on angle $\omega_3 t$ – (where t is the time of flight from the initial point) in the direction of the rotaion of earth, i.e., eastward. Consequently, under rocket (on the same radius with it) will be the point with the same latitude $\varphi_r = \varphi_r'$ but with the longitude $\lambda = \lambda' - \omega_3 t$. Thus the geographic coordinates of the rocket in relative motion can be defined by formulas

$$\sin\varphi_r = \sin\varphi_{r\text{н}}\cos\beta' + \cos\varphi_{r\text{н}}\sin\beta'\cos\psi_\text{н}', \tag{20.3}$$

$$\sin(\lambda - \lambda_\text{н} + \omega_3 t) = \frac{\sin\beta'\sin\psi_\text{н}'}{\cos\varphi_r}. \tag{20.4}$$

By the geographic coordinates it is easy to find the central angle β passed by the rocket in relative motions and the angle ψ_H formed by the plane of meridian with the plane passing through the center of earth, point H, and the rocket:

$$\cos\beta = \sin\varphi_{r\text{н}}\sin\varphi_r + \cos\varphi_{r\text{н}}\cos\varphi_r\cos(\lambda - \lambda_\text{н}), \tag{20.5}$$

$$\sin\psi_\text{н} = \frac{\cos\varphi_r\sin(\lambda - \lambda_\text{н})}{\sin\beta}. \tag{20.6}$$

The absolute azimuth at point P will be determined from the formula

$$\sin\psi' = \frac{\cos\varphi_{r\text{н}}\sin\psi_\text{н}'}{\cos\varphi_r}. \tag{20.7}$$

Having values v', ϑ' and ψ' for absolute motion at a fixed point, the geographic coordinates of which φ_r and λ are calculated by the formulas (20.3) and (20.4), according to formulas similar to (17.8) and (17.9) we can determine the parameters v, ϑ and ψ relative to the motion for this point in the terrestrial system of coordinates, taking into account that

$$v_\text{н} = v_\text{н}',$$

$$v_\text{в} = v_\text{в}' - \omega_3 r_p\cos\varphi_r,$$

$$v_\text{в} = v_\text{в}.$$

PART TWO

BALLISTIC CALCULATIONS OF CONTROLLED LONG-RANGE ROCKETS

CHAPTER VI

METHODS OF DESIGN CALCULATION OF FLYING CHARACTERISTICS

§ 21. Classification of Ballistic Calculations

At different stages of the development of new models of long-range guided rockets ballistics has before it different problems in accordance with which all ballistic calculations can be classified.

In the beginning of sketch designing there are conducted the so-called design calculations, having as their purpose the determination of limits of values of basic design parameters of a rocket according to assigned operational requirements (OR) and the selection of the most advantageous values of these parameters taking into account all real conditions. These parameters are subsequently initial in the designing of the rocket (design, selection of power scheme of the rocket, etc.).

Depending upon the assigned requirements with the design calculations designs there can be encountered the most various problems, but most frequently the problem appears by definition of basic design parameters of the rocket corresponding to the greatest sighting range, payload weight and accuracy of hit assigned the OR. Its solution, as a rule, is accompanied by an investigation of the influence of different parameters on flying-technical characteristics.

Thus the method of design ballistic calculations should allow, not resorting to cumbersome calculations and numerical integration, a rapid determination of flying characteristics of the rocket according to its design parameters and, inversely, according to the assigned flying-tactical characteristics the design parameters of the rocket.

This method should also allow estimation of the influence of a change in the basic design parameters on flying-tactical characteristics of the rocket in order to enable the selection of the most advantageous combination of these parameters. The accuracy of design ballistic calculations has presented to it nonrigid requirements. An accuracy of 2-5% in the direction of a decrease in flying characteristics is fully established by project originators.

In conclusion of the sketch designing there is conducted a checking and design ballistic calculation whose purpose is a more precise definition of flying-tactical characteristics of the rocket with parameters of the rocket obtained as a result of sketch designing, and a check of their conformity to the assigned operational requirements.

As a result of this calculation in the first approximation of the form of the trajectory of the powered section should be selected, and basic initial data necessary for calculations of construction for strength (extrenal loads) and initial data necessary for calculations of the stability of flight and development of control equipment should be determined.

With these same calculations an estimate should be produced of the accuracy of the hit with the selected principles of control, and there should be produced requirements for control equipment and propulsion system, which must be observed for the execution of OR according to the accuracy of hitting.

In the process of further designing of a rocket and the manufacture of experimental models there are produced more precise definitions of check calculations according to the more precise definition of initial data. Both in design and check ballistic calculations the influence of the rotation of the earth is not considered, since project originators and designers are interested in the mean value of flying characteristics of the rocket.

Prior to the moment of plant flight tests of the first experimental models of long-range rockets there should be complied preliminary tables of firing containing in the first approximation the dependences of sighting data of rockets from the distance of firing (for defined coordinates of the place of launch and direction of firing corresponding to the selected proving ground).

These tables are complied for the whole assigned OR of the firing range on the basis of definitized ballistic calculations, taking into account peculiarities of the control system and the use of definitized initial data, obtained by calculation and experimental means (actual weight of construction and position of the center of gravity, experimental aerodynamic properties, test engine performance, experimental parameters of control equipment, etc.). At the same stage calculation of dispersion is produced.

The next stage of ballistic calculations is the compilation of tables of sighting firing according to data of experimental and special shootings. The basic requirement for the calculation of tables of sighting firing is the increase in accuracy of calculations up to such a degree that there be removed systematic diveragences between calculation data and data of firings at any distances and under conditions of firing.

Methods of the design calculation of flying characteristis of rocket expounded in this chapter are applied in the first stage of sketch designing, i.e., for the purpose of determination, as was indicated above, of limits of values of basic design parameters of the rocket.

Proceeding from the reduced requirements for the accuracy of such calculations, we will examine some of the possible diagrams of construction of the approximation method.

§ 22. Approximation Method of the Determination of Speed

In § 19 it was shown that the full flying range is determined by four kinematic parameters at the time of the turning off of the engine, for example, speed, angle of its inclination to the local horizon, altitude and distance from the point of launch, i.e.,

$$L=f(v_k, \vartheta_k, h_k, l_k). \qquad (22.1)$$

Decisive among these parameters is the speed at the end of the powered section. The angle of inclination of the tangent to the trajectory ϑ is rather rigidly joined with speed by conditions of providing the ultimate range, and therefore there is no independent value in the majority of design problems.

The powered-flight trajectory in its extent consists of a small part of the full flying range (4-10%). Therefore, even rather considerable errors in coordinates of the end of the powered section cannot have great influence on the full flying range. Subsequently we will show that it is possible to be limited by very simple graphic dependences, allowing consideration of part of the powered section. Thus the basic should be given to the determination of speed at the end of the powered section.

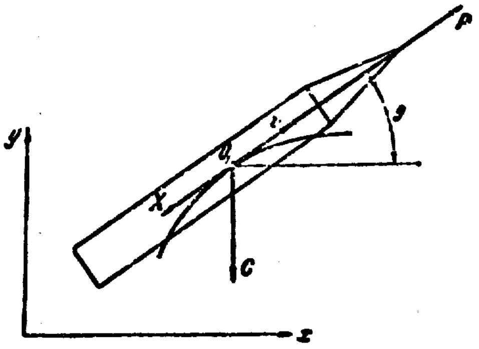

Fig. 22.1

Let us use equations of motion of the rocket as a material particle which consider only the basic forces effective in flight (Fig. 22.1):

$$\frac{dv}{dt}=\frac{P-X}{m}-g\sin\theta, \qquad (22.2)$$

$$\frac{dx}{dt}=v\cos\theta, \qquad (22.3)$$

$$\frac{dy}{dt}=v\sin\theta. \qquad (22.4)$$

Here m — the mass of the rockets; P — thrust; g — acceleration of gravity; X — drag; θ — angle of inclination of the tragent to the trajectory with respect to the horizon (angles of attack are disregarded).

For thrust there is taken the following lay of change with altitude:

$$P=P_0+S_a(p_0-p). \qquad (22.5)$$

Let us note that thrust attains a maximum value when p = 0, i.e., in a vacuum:

$$P_п=P_0+S_a p_0. \qquad (22.6)$$

Drag is determined by expression

$$X=\frac{\rho v^2}{2}c_x S. \qquad (22.7)$$

Let us write equation (22.2) in a somewhat different form:

$$dv=\left(\frac{P}{m}-g\sin\theta-\frac{X}{m}\right)dt \qquad (22.8)$$

and let us introduce the following designations:

$$\mu=\frac{m}{m_0}, \qquad (22.9)$$

$$T=\frac{m_з}{\dot{m}}\,[\,s\,], \qquad (22.10)$$

$$a_0'=\frac{P_0}{\dot{m}}\left[\frac{m}{s}\right], \qquad (22.11)$$

$$a_п'=\frac{P_п}{\dot{m}}\left[\frac{m}{s}\right], \qquad (22.12)$$

$$p_м=\frac{G_0}{S}\left[\frac{kgf}{m^2}\right], \qquad (22.13)$$

$$P_{уд0}=\frac{P_0}{\dot{G}}\left[\frac{kgf}{kgf/s}\right]. \qquad (22.14)$$

$$P_{\text{уд.0}} = \frac{P_0}{G}\left[\frac{\text{kgf}}{\text{kgf/s}}\right]. \tag{22.15}$$

where from (22.9) and (22.10 ensue the relations

$$\mu = \frac{m}{m_0} = \frac{m_0 - \dot{m}t}{m_0} = 1 - \frac{t}{T}, \tag{22.16}$$

$$t = T(1 - \mu), \tag{22.17}$$

$$T = \frac{t}{1 - \mu}. \tag{22.18}$$

In these expressions:

μ is the dimensionless coefficient characterizing the relative weight of the rocket, i.e., showing what part of the initial weight is maintained by the rocket at an examined moment. Coefficient μ theoretically can be changed from 1 to 0. At the time of launch $\mu = 1$, at the time of the turning off of the engine μ takes the minimum value for the given trajectory $\mu_к$. Quantity $\mu_к$ to a known degree characterizes the perfection of construction. Quantity $(1 - \mu)$ shows that part of the initial weight is expanded to the examined moment.

T is the ideal time, i.e., the time of operation of the engine of such an "ideal" rocket, for which the final value $\mu_к = 0$. In other words, T is the time during which at a given constant flow rate per second a quantity of fuel equal in weight to the initial weight of the rocket would burn. Quantities T and t are connected with each other by defined dependences (22.16), (22.17) and (22.18).

u'_0 is the fictitious exhaust velocity of combustion products on land and is calculated as the ratio of absolute thrust for on land (after subtracting losses to control) to the flow rate per second of mass.

$u'_п$ is the fictitious exhaust velocity of combustion products and in a vacuum is calculated as the ratio of absolute thrust in a vacuum (after subtracting losses against control) to the flow rate per second of mass.

Neither u'_0 nor $u'_п$ are the true exhaust velocity of gases from the nozzle, which practically does not depend on the altitude of flight of the rocket. The fictitious exit velocities[1] u'_0 and $u'_п$ physically mean the quantity of absolute thrust after subtracting losses to control, arriving on each unit of flow rate per second of mass.

By flow rate per second of mass is meant to total flow rate of all components participating in a decrease in weight of the rocket. Its change with the course of time will be disregarded.

$p_м$ is the initial load on the middle section of the transverse load, i.e., the initial weight arriving per unit area of the largest cross section of the rocket.

$P_{\text{уд0}}$ – specific thrust on earth.

---

[1] Subsequently instead of the term "fictitious exit velocity," for brevity we will use the term "exit velocity."

$P_{\text{уд.п}}$ – specific thrust in a vacuum.

From (22.11) and (22.14) we can obtain:

$$m_0 g\dot{\mu} = P_{\text{уд}0}\dot{G}.$$

or

$$w_0' = gP_{\text{уд}0} \qquad (22.19)$$

and, analogously,

$$w_{\text{п}}' = gP_{\text{уд.п}}. \qquad (22.20)$$

In expression (22.10), on the basis (22.11) and (22.19),

$$\dot{m} = \frac{P_0}{gP_{\text{уд}0}}.$$

therefore

$$T = \frac{m_0 g P_{\text{уд}0}}{P_0} = \frac{G_0 P_{\text{уд}0}}{P_0} = \nu_0 P_{\text{уд}0}\ [\text{s}]. \qquad (22.21)$$

here

$$\nu_0 = \frac{G_0}{P_0}.$$

We will transform every member of the expression (22.8) separately. First member P/m: considering (22.5), (22.6) and (22.9)-(22.12), we have

$$\frac{P}{m} = \frac{P_0 + S_a(p_0 - p)}{m_0\mu} = \frac{P_{\text{п}} - (P_{\text{п}} - P_0)\frac{p}{p_0}}{m_0\mu} =$$
$$= \frac{1}{T\mu}\left[w_{\text{п}}' - (w_{\text{п}}' - w_0)\frac{p}{p_0}\right].$$

We leave the second member in constant form. The third member X/m:

$$\frac{X}{m} = \frac{\rho\frac{v^2}{2}c_x S g}{G} = \frac{g}{p_{\text{м}}}\,\frac{\frac{\rho v^2}{2}c_x}{\mu}.$$

Inserting these values in equation (22.8), we obtain:

$$dv = \left\{\frac{1}{T\mu}\left[w_{\text{п}}' - (w_{\text{п}}' - w_0)\frac{p}{p_0}\right] - g\sin\theta - \frac{g}{p_{\text{м}}}\,\frac{\frac{\rho v^2}{2}c_x}{\mu}\right\}dt.$$

From (22.17) we have

$$dt = -T\,d\mu. \qquad (22.22)$$

Therefore, finally

$$dv = -\left[u'_a - (u'_a - u_0)\frac{p}{p_0}\right]\frac{d\mu}{\mu} + gT\sin\theta\, d\mu + \frac{gT}{p_{\text{м}}}\frac{qc_x}{\mu}d\mu,$$

where the impact pressure $\rho v^2/2$ is designated by q.

Let us integrate the equation obtained from v_0 to v and from μ_0 to μ:

$$v - v_0 = -u'_a \ln\frac{\mu}{\mu_0} + (u'_a - u_0)\int_{\mu_0}^{\mu}\frac{p}{p_0}\frac{d\mu}{\mu} +$$
$$+ T\int_{\mu_0}^{\mu} g\sin\theta\, d\mu + \frac{gT}{p_{\text{м}}}\int_{\mu_0}^{\mu}\frac{qc_x}{\mu}d\mu.$$

As the lower limit of integration we take the parameters of motion at the time of launch, i.e., $\mu_0 = 1$, $v_0 = 0$. Let us obtain

$$v = -u'_a \ln\mu - (u'_a - u_0)\int_{\mu}^{1}\frac{p}{p_0}\frac{d\mu}{\mu} -$$
$$- T\int_{\mu}^{1} g\sin\theta\, d\mu - \frac{gT}{p_{\text{м}}}\int_{\mu}^{1}\frac{qc_x}{\mu}d\mu. \qquad (22.23)$$

We will designate in equation (22.23)

$$I_1 = \int_{\mu}^{1} g\sin\theta\, d\mu, \qquad (22.24)$$

$$I_2 = \int_{\mu}^{1} qc_x\frac{d\mu}{\mu}, \qquad (22.25)$$

$$I_3 = \int_{\mu}^{1}\frac{p}{p_0}\frac{d\mu}{\mu}. \qquad (22.26)$$

Thus for the calculation of speed we will obtain the following basic expression:

$$v = -u'_a \ln\mu - TI_1 - \frac{gT}{p_{\text{м}}}I_2 - (u'_a - u_0)I_3, \qquad (22.27)$$

or, considering (22.19)-(22.21),

$$v = -gP_{\text{уд.п}}\ln\mu - \nu_0 P_{\text{уд.0}}I_1 - \frac{g\nu_0 P_{\text{уд.0}}}{p_{\text{м}}}I_2 -$$
$$- g(P_{\text{уд.п}} - P_{\text{уд.0}})I_3.$$

From (22.27) it is clear that the speed of the rocket is determined by the following basic design parameters: μ — ratio of running weight to the initial; $P_{\text{уд.п}}$ — specific thrust in a vacuum; ν_0 — ratio of initial weight to the initial thrust; $p_{\text{м}}$ — initial load on the middle section; $P_{\text{уд}0}$— specific thrust on earth, or the difference $(P_{\text{уд.п}} - P_{\text{уд}0}) = \Delta P_{\text{уд}}$, determined by the nozzle height of the engine.

In formula (22.27) the member $-u'_{\text{п}} \ln \mu$ determines the speed of the moving under a condition of the absence of attraction of the earth and influence of the atmosphere. The exit velocity of the gases (and, consequently, thrust) will in this case be constant and maximum. The member TI_1 determines the loss of speed induced by the action of gravity. This loss is most considerable among all others and should be considered first. The third component $\frac{gT}{p_{\text{м}}} I_2$ is a loss of speed for the surmounting of drag. The relative magnitude of the loss of speed to the surmounting of drag $\frac{gT}{p_{\text{м}}} I_2/v$ is less the more powerful the rocket. Being important factor in the determination of the speed of small rockets, this loss gradually decreases, consisting for long-range rockets 2-3% and even less.

Since the rocket moves in an atmosphere where the pressure of the atmosphere varies with altitude, then the thrust according to (22.5) will be variable, being increased from a minimum ground value up to a maximum in a vacuum. Therefore, the product $-u'_{\text{п}} \ln \mu$ gives an over estimated value of speed. The last member of equation (22.27) $(u'_{\text{п}} - u'_0) I_3$ represents a corresponding correction considering this circumstance.

If all the characteristics of rocket are assigned, the calculation of the first member of equation (22.27) causes no difficulties. Calculation of the second member of equation (22.27) is connected with the determination of the numerical value I_1. For this it is necessary to know g and $\sin \theta$ in function μ or t.

In the first approximation and with sufficient basis (altitudes on the powered section as compared to the radius of earth are small), it is possible to consider $g = \text{const}$. However, it is impossible not to consider angle θ constant, neither to take it as some mean value, not risking the making of a gross error. At the same time it is known that the dependence $\theta = \theta(\mu)$, selected taking into account real limitations, for all long-range rockets has approximately the same character. It is also known that small changes in the dependence $\theta = \theta(\mu)$ influence insignificantly the terminal velocity.

Therefore, wishing to free from the great number of variations of the dependence $\theta = \theta(\mu)$ and thereby facilitate calculation and make them applicable for a more general case, it is expedient to take for all trajectories a single dependence. If the accepted dependence after appropriate calculations and comparisons with more exact methods will show satisfactory accuracy, then it can be used in all further calculations without change. Such a dependence can be accepted in the form of a curve, on which the following conditions are superimposed.

Prior to the moment of time t_1, corresponding to the beginning of the curvilinear flight (beginning of the "program"), the angle $\theta = 90^\circ$. The value of the relative weight μ at this instant will be designated μ_1.

The necessary final angle of inclination of tangent to the trajectory is reached at the time t_2 corresponding to $\mu = \mu_2$. At this point the derivative of angle θ in time (and at μ) is equal to zero. In the interval between $\mu = \mu_1$ and $\mu = \mu_2$ the angle θ is changed by a square parabola.

After $\mu = \mu_2$ the angle of inclination of the tangent to the trajectory remains constant prior to the moment of turning off of the engine.

It is convenient to record equation of parabola in the form

$$\theta = A(\mu - \mu_2)^2 + B(\mu - \mu_2) + C. \tag{22.28}$$

Coefficient A, B and C are easily determined from shown conditions. So that the problem is more concrete, it is necessary to assign defined value μ_1 and μ_2, constants for all possible cases of calculation. It is possible to consider established that the vertical section continues up to values of μ close to 0.95. Therefore, it is quite natural to assume $\mu_1 = 0.95$.

Further, the powered sections of almost all long-range rockets possess that property which after $\mu = 0.4$-0.5 the trajectory is either rectilinear or very closely approaches a straight line. At the same time values of μ_K greater then 0.3-0.4, as a rule, are not encountered. Proceeding from this, it is possible to consider sections of the trajectory after $\mu = 0.45$ for all rockets rectilinear and differing from each other only by a value of the angle of inclination, so that $\mu_2 = 0.45$.

Inasmuch as it is profitable to conduct firing at optimum angles, then θ_K for various trajectories will be different. Thus losses of speed from gravity will be a function of the final angle θ_K and μ.

The dependence $\theta = \theta(\mu)$ with parameter θ_K satisfying all the conditions, has the form

$$\left.\begin{aligned} &\theta = 90^\circ && \text{when} \quad 1 > \mu > 0.95, \\ &\theta = 4\left(\frac{\pi}{2} - \theta_K\right)(\mu - 0.45)^2 + \theta_K && \text{when} \quad 0.95 > \mu > 0.45, \\ &\theta = \theta_K && \text{when} \quad 0.45 > \mu. \end{aligned}\right\} \tag{22.29}$$

The dependence $\theta = \theta(\mu)$ is represented graphically on Fig. 22.2. Values of the

$$I_1 = \int_{\mu}^{1} g \sin\theta \, d\mu$$

are given in Table 22.1 and on Fig. 22.3, which one should use in carrying out the concrete calculations.

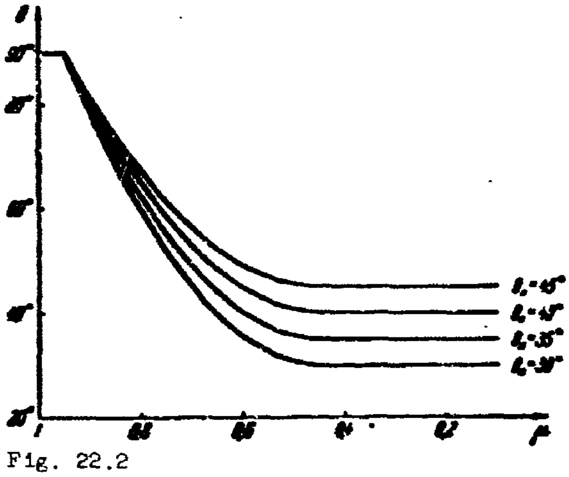

Fig. 22.2

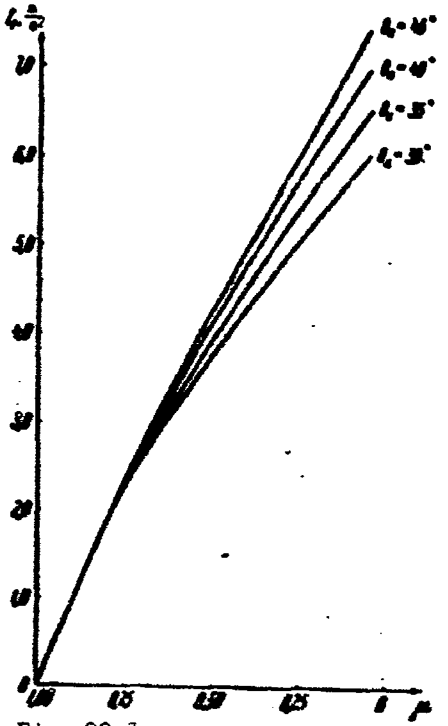

Fig. 22.3

Table 22.1

μ	− ln μ	$l_s \frac{\pi}{2}$			
		$\theta_s = 30°$	$\theta_s = 35°$	$\theta_s = 40°$	$\theta_s = 45°$
1,00	0,0000	0,000	0,000	0,000	0,000
0,90	0,1054	0,978	0,976	0,979	0,979
0,80	0,2231	1,887	1,899	1,910	1,919
0,70	0,3567	2,656	2,700	2,741	2,779
0,60	0,5108	3,283	3,381	3,472	3,555
0,50	0,6931	3,911	3,976	4,130	4,272
0,45	0,7985	4,058	4,259	4,447	4,620
0,40	0,9163	4,303	4,540	4,762	4,967
0,35	1,0498	4,548	4,822	5,077	5,313
0,30	1,2040	4,794	5,103	5,392	5,663
0,28	1,2730	4,892	5,215	5,519	5,799
0,26	1,3471	4,990	5,328	5,645	5,938
0,24	1,4271	5,088	5,441	5,771	6,076
0,22	1,5141	5,186	5,553	5,897	6,215
0,20	1,6094	5,284	5,666	6,023	6,354
0,19	1,6607	5,333	5,722	6,086	6,423
0,18	1,7148	5,382	5,778	6,149	6,493
0,17	1,7720	5,431	5,834	6,212	6,562
0,16	1,8326	5,480	5,891	6,275	6,631
0,15	1,8971	5,529	5,947	6,338	6,701
0,14	1,9661	5,579	6,003	6,401	6,770
0,13	2,0402	5,628	6,060	6,464	6,839
0,12	2,1203	5,677	6,116	6,527	6,909
0,11	2,2073	5,726	6,172	6,591	6,978
0,10	2,3026	5,775	6,228	6,654	7,048
0,09	2,4079	5,824	6,285	6,717	7,117
0,08	2,5257	5,873	6,341	4,780	7,186
0,07	2,6593	5,922	6,397	6,843	7,256
0,06	2,8134	5,971	6,453	6,906	7,325
0,05	2,9957	6,020	6,510	6,969	7,394

Integral I_2 expresses influence of drag:

$$I_2 = \int_{\mu}^{1} \frac{q c_x}{\mu} d\mu.$$

In order to calculate this integral, it is necessary beforehand to know $q = \frac{\rho v^2}{2}$ and $c_x(M, h)$ in function μ. To obtain these dependences we will use the first two members of equation (22.27), giving speed in function μ under the condition of the absence of the atmosphere.

Let us denote

$$v_1 = -u'_{\text{п}} \ln \mu - T I_1. \qquad (22.30)$$

Then the altitudes corresponding to these speeds will be

$$y_1 = \int_0^t v_1 \sin \theta \, dt = T \int_{\mu}^{1} v_1 \sin \theta \, d\mu. \qquad (22.31)$$

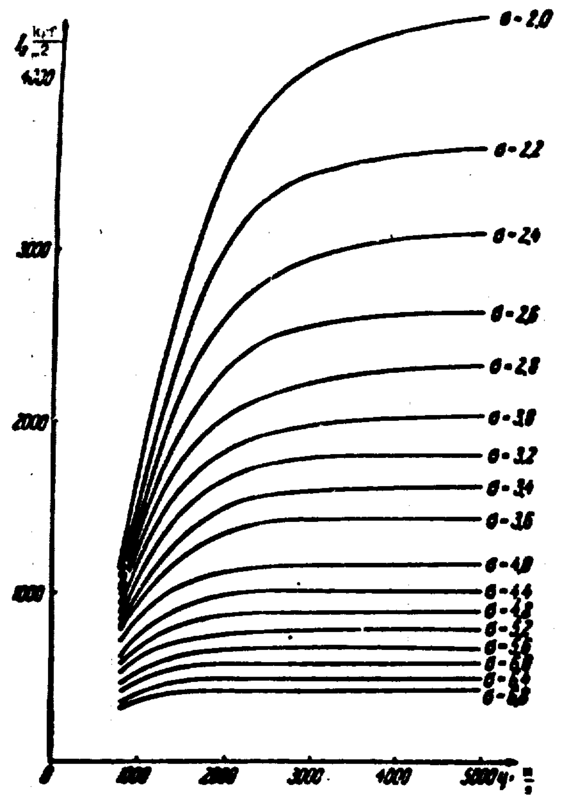

Fig. 22.4

Let us call v_1 and y_1 the speed and altitude of the first approximation. Knowing v_1 and y_1, it is easy to calculate

$$I_2 = \int_{\mu}^{1} \frac{\rho v^2}{2} c_x \frac{d\mu}{\mu}.$$

During calculation of I_2, instead of v we will insert v_1; ρ will be taken not from the true value of y but y_1; c_x will also be determined with respect to v_1 and y_1.

The great quantity of calculations conducted for the purpose of determination of I_2, permitted establishing the following empirical dependence: values of the integral corresponding to the same value of speed, plotted on the graph depending upon quantity[1]

$$\sigma = T \sqrt{u'_{\text{cp}}} \sin \theta_{\text{к}} \cdot 10^{-3}. \qquad (22.32)$$

where $u'_{cp} = (u' + u'_0)/2$, have insignificant scattering around a certain mean curve. Therefore, it was found possible to construct the dependence $I_2 = f(v_1)$ with parameter σ (Fig. 22.4 and Fig. I of the Appendex[2]). Values of I_2 are obtained quite close to the real, since speeds during calculations of I_2 are taken overestimated (from the first approximation), and the densities understated (because of overestimated altitudes of the first approximation).

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[1] With the calcuation of σ by the formula (22.32) it is necessary to take T per s, and u'_{cp} m/s.

[2] In the Appendex graphs are given which can be used for design calculations.

Thus for the determination of the loss of speed on drag it is sufficient to calculate

$$\sigma = T\sqrt{u'_{cp}}\sin\theta_к \cdot 10^{-3}$$

and, taking it as a parameter, to find I_2 depending upon speed v_1 known beforehand from the first approximation. Obtained I_2 must then be multiplied by quantity $\frac{gT}{p_м}$, which characterizes each rocket taken separately. Hence it is clear that the loss of speed for overcoming drag depends on the transverse load. The greater the transverse load, the less the loss of speed during the passage of the rocket through the atmosphere. Therefore it is desirable to have p_M as large as possible not causing damages, however, to other flying characteristics of the rocket.

It is necessary to note that with calculation of I_2 for all rockets identical coefficients of c_x were accepted. This circumstance, however, does not lead to considerable errors according to the following causes:

1) for all rockets of normal ballistic scheme coefficients c_x are approximately identical;

2) dependences $c_x(M, h)$, on which are conducted exact calculations for concrete rockets, themselves possess considerable errors;

3) the influence of drag, in general, is small, especially for powerful rockets designed for firing at great distances. Therefore, the error owing to c_x has insignificant influence.

The last correction $(u'_п - u'_0)I_3$ considers the change in thrust with altitude. For the calculation of I_3 it is necessary to know the dependence $\frac{p}{p_0} = f(\mu)$, this can be known if the altitude y in function μ is known. There were conducted a great number of calculations for the purpose of the determination of I_3, where altitudes y were taken from the second approximation. We call the altitude of the second approximation the altitude obtained during integration of equation (22.4), in which speed is determined by the formula (22.27) taking into account the first three members, i.e.,

$$y_2 = \int_{\mu}^{1}\left[-u'_к \ln\mu - TI_1 - \frac{gT}{p_м}I_2\right]T\sin\theta\, d\mu. \qquad (22.33)$$

As a result of processing these calculations we succeeded for quantity η equal to[1]

$$\eta = 0.001u'_{cp}\sqrt[3]{u'_{cp}\sin\theta_к}\, I_3.$$

in establishing the empirical dependence on the parameter ν_0 and time of flight t.

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[1] In the calculation of η it is necessary to take u'_{cp} in m/s.

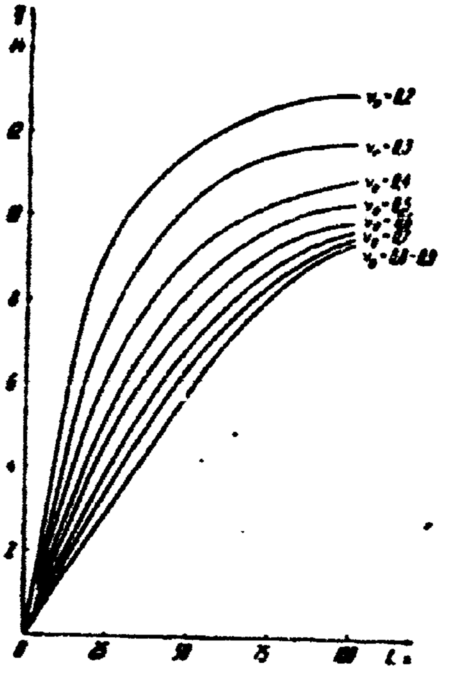

Fig. 22.5

This dependency is depicted on Fig. 22.5 and Fig. II of the Appendex.

Parameter ν_0 is calculated by one of three formulas:

$$\nu_0 = \frac{gT}{u_0'} = \frac{T}{P_{уд0}} = \frac{G_0}{P_0}.$$

Thus, having a concrete rocket, we calculate for it ν_0 and find the parameter η the the moment of time interesting to us. Then we calculate I_3 by the formula

$$I_3 = \frac{\eta}{0.001 a_{cp}' \sqrt{a_{cp}' \sin\theta_к}}. \qquad (22.34)$$

The product $I_3(u_п' - u_0')$ gives the unknown loss of speed for overcoming the counterpressure of air.

After determination of all the losses we calculate formula (22.27) the final speed. Fig. 22.6 gives curves allowing on a particular example[1] to trace the change of speed depending upon μ and the relation between separate members of formual (22.27). Plotted on this graph downwards are losses of speed Δv_1, Δv_2 and Δv_3, referred to the true speed of v.

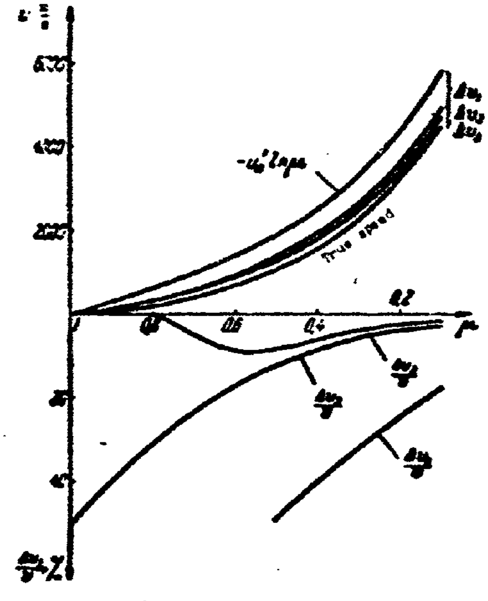

Fig. 22.6

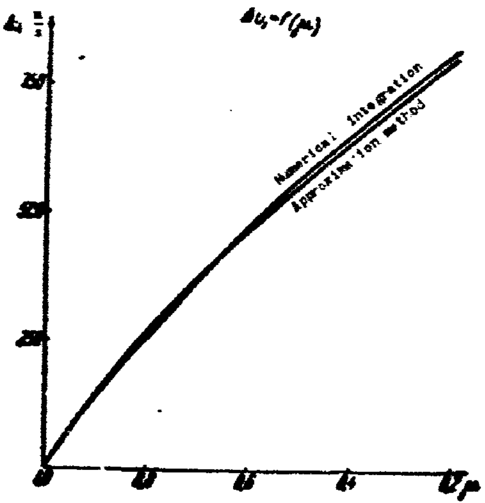

Fig. 22.7

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[1] With the following initial conditions: $\nu_0 = 0.577$; $P_{уд.п} = 288\ \frac{\mathrm{kg}}{\mathrm{kg/s}}$; $P_{уд0} = 240\ \frac{\mathrm{kg}}{\mathrm{kg/s}}$; $p_м = 10{,}000\ \frac{\mathrm{kg}}{\mathrm{m^2}}$; $\theta_к = 38^\circ 20'$.

Here there are designated: $\Delta v_1 = TI_1$ — loss of speed for overcoming gravity; $\Delta v_2 = \frac{gT}{p_a} I_2$ — loss of speed for overcoming drag; $\Delta v_3 = (u_n' - u_0')I_3$ — loss of speed for overcoming counterpressure of the atmosphere.

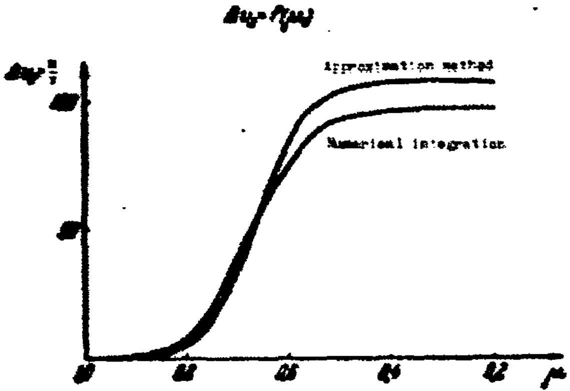

Fig. 22.8

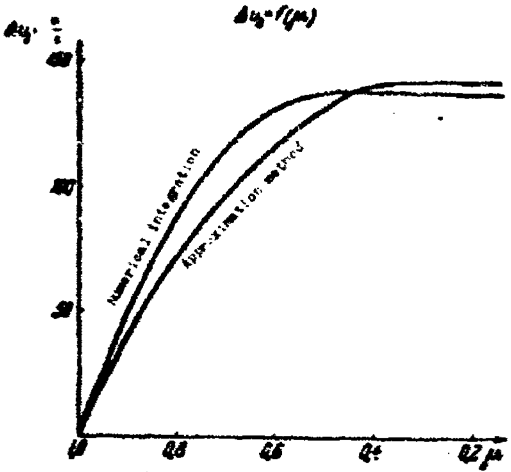

Fig. 22.9

Fig. 22.7, 22.8, 22.9 give a comparison of separate losses (Δv_1, Δv_2, Δv_3) calculated for the same example by two methods, namely, numerical integration of the system (22.2)-(22.4) and the method just now expounded.

§ 23. Determination of Full Range

For purposes of designing it is possible to propose simple dependences which enable the possibility of finding the full flying range in a function of the speed attained up to moment of turning off of the engine or, conversely, the speed necessary for achievement of the assigned distance, without recourse to the calculation of coordinates of end of the powered section. The proposed formulas are not exact, but for the first approximation they give quite satisfactory results.

Let us express the full range as

$$L = kL_{эл}. \qquad (23.1)$$

Here by $L_{эл}$ is meant the range concluded between the two radii MK and MN, conducted from the center of the earth and intersecting the trajectory on its ascending and and descending phases at an altitude of end of the powered section (Fig. 23.1).

Thus coefficient k expresses the ratio of full range to its purely elliptic part determined by arc ED along the surface of the earth. We use only conditionally the term "purely elliptic part of the trajectory," understanding by this only the fact that the influence of the atmosphere affects the flight negligibly. We determine the flying range corresponding to this section of the trajectory by the formulas

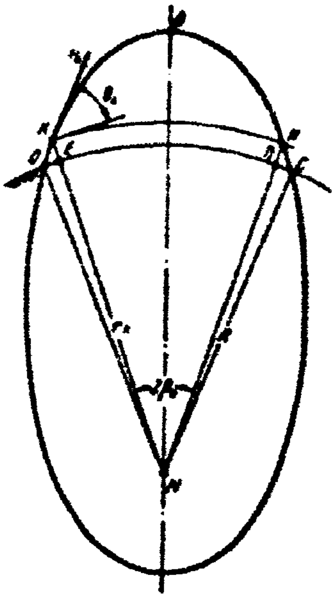

Fig. 23.1

of elliptic theory, about which was discussed in § 19.

Of all the diagram of calculations examined for this case we will use only the one corresponding to the case of firing at an optimum angle (see (19.31) and (19.33)):

$$\mathrm{tg}\,\beta_k = \frac{\nu_k}{2\sqrt{1-\nu_k}}. \qquad (23.2)$$

whence, considering that $L_{эл} = 2R\beta_B$,

$$L = kL_{эл} = 2kR\,\mathrm{arctg}\,\frac{\nu_k}{2\sqrt{1-\nu_k}}. \qquad (23.3)$$

Here[1]

$$\nu_k = \frac{v_k^2 r_k}{398\,620}.$$

If one were to take $r_K = R$, then $\nu_K = \frac{v_k^2}{62{,}57}$ Substituting this value ν_K into (23.3), we will obtain

$$L = 2kR\,\mathrm{arctg}\,\frac{v_k^2}{15{,}82\sqrt{62{,}57 - v_k^2}}. \qquad (23.4)$$

If the arc $\mathrm{arctg}\,\frac{v_k^2}{15{,}82\sqrt{62{,}57 - v_k^2}}$ is taken in degrees, then

$$L = 222{,}4k\,\mathrm{arctg}\,\frac{v_k^2}{15{,}82\sqrt{62{,}57 - v_k^2}}. \qquad (23.5)$$

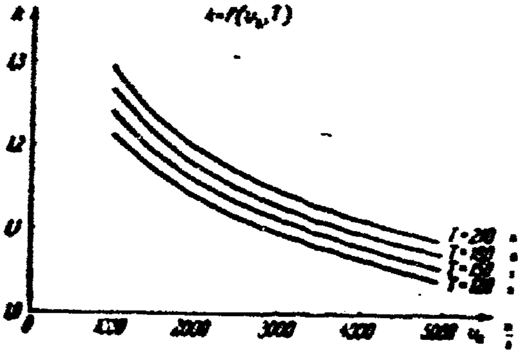

Fig. 23.2

The value of coefficients k is not identical for different rockets but depends on the speed or range and on basic design parameters, in the first place on those

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[1] In this and subsequent formulas it follows to express v_K in km/s, and r_K, R and L in km.

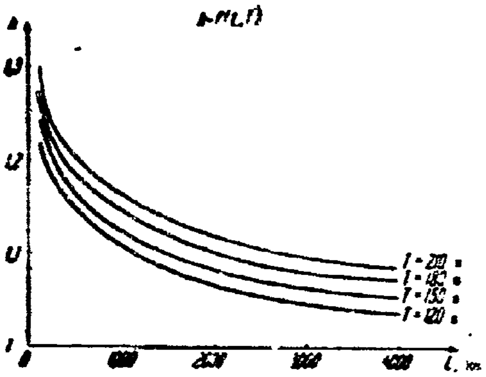

Fig. 23.3

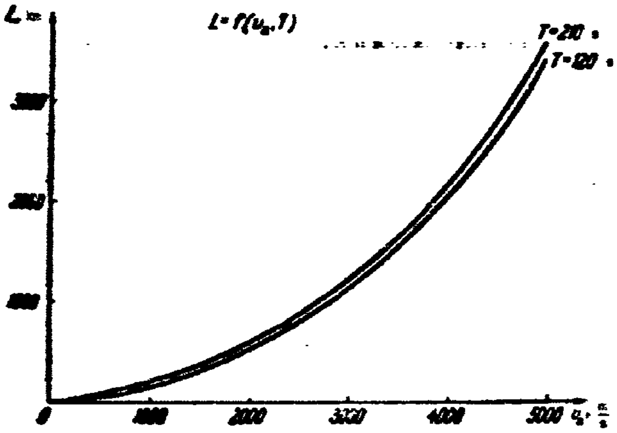

Fig. 23.4

by which is determined duration, more correctly the extent of the powered section. Such parameters are $P_{уд0}$ and ν_0, the influence of which in totality can be replaced by value $T = \nu_0 P_{уд0}$.

To determine k on Fig. 23.2 (Fig. III of the Appendex) there is the curve $k = k(v_к)$ with parameter T plotted on the basis of analysis of a large quantity of accurate calculations.

If we copy formula (23.5) with respect to $v_к$, then we will find

$$v_к = 11.19 \sqrt{\operatorname{tg} \frac{L}{222.4k} \operatorname{tg}\left(45° - \frac{L}{2 \cdot 222.4k}\right)}. \quad (23.6)$$

In (23.6) the quantity L/222.4k is measured in degrees. To determine k depending upon the range on Fig. 23.3 (Fig. IV of the Appendex) there is plotted the curve $k = k(L)$ with parameter T. By the formulas (23.5) and (23.6) calculations are performed, and on Fig. 23.4 (Fig. V of the Appendex) there is plotted the dependence $L = f(v_к)$, using which it is possible to solve the direct and inverse problem of ballistics.

§ 24. Final Formulas for a Rough Estimate of Flying Range

In design ballistic calculations it is sometimes found useful to establish at least a very approximate dependence of flying characteristics of a rocket from its basic design parameters expressed by means of final formulas. Such a dependence can be obtained by examining the motion of a rocket under the action of only two basic forces: thrust P and weight G, considering that the acceleration of gravity g has the same value (in magnitude and the direction) at all points of space.

Let us direct the x axis of the rectangular system of coordinates on the tangent to the surface of the earth at the launch point and the y axis vertically upwards. We will consider that at the initial moment (when t = 0) the coordinates and speed of the rocket are equal to zero. Motion will be considered flat; this is as it should be, if besides the assumptions made we demand that the tractive force during the whole time of operation of the engine act in one plane. The angle which this force composes with the plane of the horizon (with x axis) will be designated φ. Then the equations of motion of the rocket will have the form

$$m\frac{d^2x}{dt^2}=P\cos\varphi,$$

$$m\frac{d^2y}{dt^2}=P\sin\varphi-G,$$

or

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=gn\cos\varphi,\\ \frac{d^2y}{dt^2}&=g(n\sin\varphi-1).\end{aligned}\right\}\qquad(24.1)$$

where n designates the ratio P/G (G-forces), which we will consider by the known function of time n = n(t).

Integrating twice equations (24.1), we obtain:

$$\left.\begin{aligned}\frac{dx}{dt}&=g\int_0^t n(\tau)\cos\varphi(\tau)\,d\tau,\\ \frac{dy}{dt}&=g\left(\int_0^t n(\tau)\sin\varphi(\tau)\,d\tau-t\right)\end{aligned}\right\}\qquad(24.2)$$

and

$$\left.\begin{aligned}x&=g\int_0^t\int_0^{\tau_1} n(\tau)\cos\varphi(\tau)\,d\tau\,d\tau_1,\\ y&=g\left(\int_0^t\int_0^{\tau_1} n(\tau)\sin\varphi(\tau)\,d\tau\,d\tau_1-\frac{t^2}{2}\right).\end{aligned}\right\}\qquad(24.3)$$

In the obtained double integrals the region of integration in the plane of variables τ and τ_1 constitutes of triangle determined by inequalities

$$0<\tau<\tau_1<t.$$

Changing the order of integration, we obtain

$$x=g\int_0^t\int_\tau^t n(\tau)\cos\varphi(\tau)\,d\tau_1\,d\tau=g\int_0^t(t-\tau)\,n(\tau)\cos\varphi(\tau)\,d\tau$$

and, analogously,

$$y = g\left(\int_0^t (t-\tau)\,n(\tau)\sin\varphi(\tau)\,d\tau - \frac{t^2}{2}\right).$$

Let us assume that the time of operation of the engine (duration of the powered section) is equal to t_K, so that

$$n(t) > 0 \text{ when } 0 < t < t_K$$

and

$$n(t) = 0 \text{ when } t > t_K.$$

Then for $t \geq t_K$ these formulas will be correct:

$$\left.\begin{aligned} v_x &= g\int_0^{t_K} n\cos\varphi\,d\tau,\\ v_y &= g\left(\int_0^{t_K} n\sin\varphi\,d\tau - t\right); \end{aligned}\right\} \tag{24.4}$$

$$\begin{aligned} x &= g\int_0^{t_K} (t-\tau)\,n\cos\varphi\,d\tau,\\ y &= g\left(\int_0^{t_K} (t-\tau)\,n\sin\varphi\,d\tau - \frac{t^2}{2}\right). \end{aligned}$$

where n and φ integrand expressions are functions of τ. These formulas can be used, in particular, for the determination of coordinates of the impact point, substituting in them instead of t, the time of flight prior to the encounter with the surface of earth t_C (assuming, of course, that $t_C > t_K$):

$$\left.\begin{aligned} v_{xC} &= g\int_0^{t_K} n\cos\varphi\,d\tau,\\ v_{yC} &= g\left(\int_0^{t_K} n\sin\varphi\,d\tau - t_C\right); \end{aligned}\right\} \tag{24.5}$$

$$\left.\begin{aligned} x_C &= g\int_0^{t_K} (t_C-\tau)\,n\cos\varphi\,d\tau,\\ y_C &= g\left(\int_0^{t_K} (t_C-\tau)\,n\sin\varphi\,d\tau - \frac{t_C^2}{2}\right). \end{aligned}\right\} \tag{24.6}$$

To determine the complete time of flight t_C it is necessary to insert expressions for x_C and y_C into equation of the surface of earth

$$y = f(x) \tag{24.7}$$

and solve the equation with respect to t_C. We will not specify as yet the form of function $f(x)$ in equation (24.7).

It is natural to approach the obtaining of the ultimate range of flight of the rocket. Therefore, we will solve the following variational problem: what should function $\varphi = \varphi(\tau)$ be so that the abscissa x_C of the point of impact is maximum. For this we will find variations Δx_C and Δy_C owing to the variation $\delta\varphi$.

As usual, we can write:

$$\left.\begin{aligned} \Delta x_C &= \delta x_C + v_{xC}\,\Delta t_C, \\ \Delta y_C &= \delta y_C + v_{yC}\,\Delta t_C, \end{aligned}\right\} \tag{24.8}$$

where Δt_C is the variation of time of the flight and δx_C and δy_C are variations of coordinates at the fixed value $t = t_C$. These variations are equal

$$\left.\begin{aligned} \delta x_C &= -g\int_0^{t_C}(t_C-\tau)\,n\sin\varphi\,\delta\varphi\,d\tau, \\ \delta y_C &= g\int_0^{t_C}(t_C-\tau)\,n\cos\varphi\,\delta\varphi\,d\tau. \end{aligned}\right\} \tag{24.9}$$

Excluding from equations (24.8) and (24.9) quantities Δt_C, δx_C and δy_C, we will obtain

$$\begin{aligned} -v_{yC}\,\Delta x_C + v_{xC}\,\Delta y_C &= \\ &= g\int_0^{t_C}(t_C-\tau)\,n\,(v_{yC}\sin\varphi + v_{xC}\cos\varphi)\,\delta\varphi\,d\tau. \end{aligned}$$

Furthermore, x_C and y_C are connected with each other by the equation of the surface of earth

$$y_C = f(x_C).$$

and therefore

$$\Delta y_C = f'(x_C)\,\Delta x_C. \tag{24.10}$$

Thus

$$\begin{aligned} [-v_{yC} + v_{xC}f'(x_C)]\,\Delta x_C &= \\ &= g\int_0^{t_C}(t_C-\tau)\,n\,(v_{yC}\sin\varphi + v_{xC}\cos\varphi)\,\delta\varphi\,d\tau. \end{aligned}$$

The necessary condition of extremum of quantity x_C is the identical conversion into zero of the first variation Δx_C, in other words, the fulfillment of equality

$$\int_0^{t_k}(t_C-\tau)n(v_{yC}\sin\varphi+v_{xC}\cos\varphi)\delta\varphi\,d\tau=0.$$

The variation $\delta\varphi$ in our formulation of the problem has no limitations on it, so that basically the lemma of calculus of variation is applicable, from which it follows that angle φ should satisfy equation

$$v_{yC}\sin\varphi+v_{xC}\cos\varphi=0. \tag{24.11}$$

at least for those values of τ, where $n \neq 0$, and only such values interest us. The value of φ satisfying condition (24.11) will be designated by φ_0. Condition (24.11) can be rewritten in the form

$$\operatorname{tg}\varphi_0=-\frac{v_{xC}}{v_{yC}}=-\operatorname{ctg}\theta_C,$$

where θ_C is the angle composed by the velocity vector at the impact point on the x axis. It follows from this:

$$\varphi_0=\theta_C\pm\frac{\pi}{2}.$$

Thus for achievement of the ultimate range of flight it is necessary that during the entire time of operation of the engine the direction of the thrust remain constant, where in such a way that the direction of speed of the rocket at the time of its encounter with the surface of earth is found perpendicular to the direction of the thrust. Let us study in greater detail this optimal state of motion of the rocket.

Let us introduce these designations:

$$\int_0^{t_k}n\,d\tau=N, \tag{24.12}$$

$$\int_0^{t_k}n\tau\,d\tau=N_1. \tag{24.13}$$

when $\varphi = \varphi_0$ = const expressions (24.5) and (24.6) can be rewritten in the form

$$\left.\begin{aligned}v_{xC}&=gN\cos\varphi_0,\\ v_{yC}&=g(N\sin\varphi_0-t_C);\end{aligned}\right\} \tag{24.14}$$

$$\left.\begin{aligned}x_C&=g(Nt_C-N_1)\cos\varphi_0,\\ y_C&=g\left[(Nt_C-N_1)\sin\varphi_0-\frac{t_C^2}{2}\right].\end{aligned}\right\} \tag{24.15}$$

We will insert expressions (24.14) for v_{xC} and v_{yC} into equation (24.11), which should satisfy the optimum angle φ_0. Let us obtain

$$g(N\sin\varphi_0-t_C)\sin\varphi_0+gN\cos^2\varphi_0=0.$$

or

$$N - t_C \sin\varphi_0 = 0. \tag{24.16}$$

whence

$$t_C = \frac{N}{\sin\varphi_0}. \tag{24.17}$$

This expression for t_C will be substituted into formulas (24.15) for coordinates of the point of impact:

$$x_C = g\left(\frac{N^2}{\sin\varphi_0} - N_1\right)\cos\varphi_0. \tag{24.18}$$

$$y_C = g\left[\left(\frac{N^2}{\sin\varphi_0} - N_1\right)\sin\varphi_0 - \frac{N^2}{2\sin^2\varphi_0}\right]. \tag{24.19}$$

Instead of quantities N and N_1 it will subsequently be more convenient to use dimensionless quantities

$$a = \frac{N_1}{N^2}. \tag{24.20}$$

$$b = \frac{gN^2}{R}, \tag{24.21}$$

where R is the radius of earth. In these designations

$$N^2 = \frac{bR}{g}, \quad N_1 = \frac{abR}{g}$$

and

$$x_C = Rb\,\mathrm{ctg}\,\varphi_0(1 - a\sin\varphi_0). \tag{24.22}$$

$$y_C = \frac{Rb}{2\sin^2\varphi_0}(2\sin^2\varphi_0 - 2a\sin^3\varphi_0 - 1). \tag{24.23}$$

The equation for angle φ_0 can be obtained as a result of the substitution of these expressions into the equation of earth's surface (24.7). If one were to consider the earth a sphere with radius R, then this equation (more exactly, the equation of a section of the earth's surface by the plane of firing) has the form

$$y = -R + \sqrt{R^2 - x^2}.$$

Instead of it we will use the approximate equation which is obtained if we decompose $\sqrt{R^2 - x^2}$ in series in powers of x/R and are limited by two members of this decomposition:

$$y = -R + R\left(1 - \frac{x^2}{2R^2}\right).$$

or

$$y = -\frac{x^2}{2R}. \qquad (24.24)$$

Derivation of this equation justifies its use with small x. However, the application of the same equation for large x is not deprived of bases. It is known that in the central field of gravity a body thrown on a tangent to the surface of earth with an initial speed $v_0 = \sqrt{gR}$ will move all the time in a circular orbit along this surface. In the examined field of gravity the body, to which at the origin of coordinates is imparted the same speed v_0 in a horizontal direction, will move, as it is easy to determine, about a parabola described by equation (24.24). Thus this parabold in a certain meaning is the analog of the surface of earth for bodies moving with an initial "circular" speed v_0.

Inserting into equation (24.24), instead of x and y expressions (24.22) and (24.23) for x_C and y_C, we obtain

$$\frac{Rb}{2\sin^2\varphi_0}(2\sin^2\varphi_0 - 2a\sin^3\varphi_0 - 1) = -\frac{Rb^2\operatorname{ctg}^2\varphi_0}{2}(1 - a\sin\varphi_0)^2.$$

or

$$2\sin^2\varphi_0 - 2a\sin^3\varphi_0 - 1 = -b\cos^2\varphi_0(1 - a\sin\varphi_0)^2.$$

Replacing $\cos^2\varphi_0$ by $1 - \sin^2\varphi_0$ and transferring all members to the right side, we reduce this equation to the form

$$a^2b\sin^4\varphi_0 + 2a(1-b)\sin^3\varphi_0 - (2 - b + a^2b)\sin^2\varphi_0 + 2ab\sin\varphi_0 + 1 - b = 0. \qquad (24.25)$$

This equation can easily be solved by a certain numerical method. There converges rather quickly for example, interational process, founded on the formula

$$\sin\varphi_0 = \left[\frac{1}{2-b}(a^2b\sin^4\varphi_0 + 2a(1-b)\sin^3\varphi_0 - a^2b\sin^2\varphi_0 + 2ab\sin\varphi_0 + 1 - b)\right]^{1/2}. \qquad (24.26)$$

ensuing from equation (24.25).

Thus, if the law of change of load factor n(t) is assigned, then formulas (24.12) and (24.13) for N and N_1, (24.20) and (24.21) for a and b, equation (24.25) for φ_0 and, finally, formulas (24.22) and (24.23) permit determining coordinate x_C and y_C of the point of impact of the rocket. Coordinate x_C can be considered an approximate value of the flying range, in any case for small distances where equation (24.24) quite well described the form of the surface of the earth.

In order to obtain the best accuracy for great distances, let us examine the limiting case of the instantaneous burning of fuel ($t_K \to 0$). In this case the integral N has the final limiting value connected with speed at the end of the powered section by the relation

$$v_K = gN$$

(see formula (24.4)). For the integral N_1, on the basis of formula (24.13), we obtain

$$N_1 = \int_0^{t_k} \varepsilon\tau\,d\tau < \int_0^{t_k} \varepsilon t_k\,d\tau = N t_k$$

and, consequently, the limiting value of N_1, and at the same time of a, is equal to zero.

Formula (24.26) gives when a = 0

$$\sin\varphi_0 = \sqrt{\frac{1-b}{2-b}},$$

after which by the formula (24.22) we obtain

$$x_C = \frac{Rb}{\sqrt{1-b}}. \tag{24.27}$$

With the instantaneous burning of fuel coordinates x_K and y_K obviously, are equal to zero. The problem of the determination of the ultimate range of flight under conditions of the elliptic theory was solved in § 19 Chapter V. For the case $r_H = r_C = R$ we obtained formula (19.33):

$$\operatorname{tg}\theta_{0\,\text{opt}} = \sqrt{1-\nu_0},$$

where

$$\nu_0 = \frac{v_K^2 r_K}{k} = \frac{(gN)^2 R}{gR^2} = \frac{gN^2}{R} = b.$$

From formulas (19.31) and (19.33) it follows:

$$\operatorname{tg}\frac{\beta_{C\,\max}}{2} = \frac{b}{2\sqrt{1-b}}.$$

Comparing this formula with formula (24.27), we obtain

$$\operatorname{tg}\frac{\beta_{C\,\max}}{2} = \frac{x_C}{2R},$$

whence, taking into account that when $x_K = y_K = 0$

$$L = l_{00} = R\beta_C,$$

we find

$$L = 2R\operatorname{arctg}\frac{x_C}{2R}. \tag{24.28}$$

This formula, which established the relation between the flying range L under conditions of the elliptic theory and coordinate x_C, calculated by the above-described method, can be expediently used when a ≠ 0. Uniting formulas (24.28) and (24.22) in one, we will obtain

$$L = 2R\operatorname{arctg}\left[\frac{b}{2}\operatorname{ctg}\varphi_0(1 - a\sin\varphi_0)\right].$$

The expounded method in pure form is too coarse for the determination of distance mainly because in it are not considered losses of speed (and consequently, distance) for overcoming drag to the motion of the rocket. However, already the formula

$$L = 0{,}75x_C$$

gives for distances up to 5000-7000 km an error not exceeding 10% with a change in design parameters of the rocket almost in the whole range of practically reasonable values.

Let us dwell now on the question of the expression of integrals N and N_1 in terms of design parameters of the rocket. Let us consider the case of the multistage rocket, consisting of m stages. Let us designate the thrust of the engine of the i-th stage by P_i, the flow rate per second of fuel on the i-th stage by $\dot{G}_i$, the initial weight of the i-th stage by G_{0i}, the time of the end of operation of the i-th and (for i < m) of the beginning of the operation of (i + 1)-th stage by $t_{\text{к}i}$. During the period of operation of the i-th stage, i.e., when $t_{\text{к}i-1} < t \leq t_{\text{к}i}$, quantities P_i and $\dot{G}_i$ will be considered constant. Then when $t_{\text{к}i-1} < t \leq t_{\text{к}i}$ for the load factor n(t) we obtain the expression

$$n(t) = \frac{P_i}{G_i} = \frac{P_i}{G_{0i} - \dot{G}_i(t - t_{\text{к}\,i-1})},$$

or

$$n(t) = \frac{P_{\text{уд}\,i}}{T_i - t},$$

where $P_{\text{уд}\,i} = \frac{P_i}{\dot{G}_i}$ is the specific thrust of the engine of the i-th stage,

$$T_i = \frac{G_{0i}}{\dot{G}_i} + t_{\text{к}\,i-1}. \tag{24.29}$$

In these designations

$$N = \int_0^{t_\text{к}} n(t)\,dt = \sum_{i=1}^{m} \int_{t_{\text{к}\,i-1}}^{t_{\text{к}\,i}} \frac{P_{\text{уд}\,i}}{T_i - t}\,dt,$$

$$N_1 = \sum_{i=1}^{m} \int_{t_{\text{к}\,i-1}}^{t_{\text{к}\,i}} \frac{P_{\text{уд}\,i}\,t}{T_i - t}\,dt.$$

But

$$\int \frac{P_\text{уд}}{T - t}\,dt = -P_\text{уд} \ln(T - t),$$

$$\int \frac{P_\text{уд}\,t}{T - t}\,dt = \int \frac{P_\text{уд}[T - (T - t)]}{T - t}\,dt = P_\text{уд}[-T \ln(T - t) - t].$$

It follows from this that

$$N = \sum_{i=1}^{m} P_{\text{уд}\,i} \ln \frac{T_i - t_{\text{к}\,i-1}}{T_i - t_{\text{к}\,i}}. \tag{24.30}$$

$$N_1 = \sum_{i=1}^{m} P_{\text{уд}\,i} \left[T_i \ln \frac{T_i - t_{\text{к}\,i-1}}{T_i - t_{\text{к}\,i}} + t_{\text{к}\,i-1} - t_{\text{к}\,i}\right]. \tag{24.31}$$

§ 25. Design Calculations with the Use of Electronic Computers

The effect from the use of electronic computers is especially great in the solution of such problems which require multiple appeal to the same algorithm, which is the most labororious part of the total calculation. Design-ballistic calculations, conducted for the purpose of selection of basic design parameters of the rocket, pertain exactly to such kind of problems.

Really, if it is required, for example, to investigate the influence on flying range of the rocket of only some three independent parameters and to give to each of these parameters at least five values, then the quantity of possible combinations of parameters will be equal to $5^3 = 125$. For each of these combinations it is necessary to select the trajectory realizing the maximum of distance. If one were to consider that five calculations for detecting the optimum trajectory are sufficient, then the total amount of calculations of trajectories will be equal to $5 \cdot 125 = 625$.

The main part of the time will be occupied by integration of the system of differential equations of motion repeated 625 times. Reducing results into a defined system convenient for analysis (grids of curves or table), it is possible to select the most profitable combination of parameters interesting to us. In certain cases on the machine can be placed the solution of an extreme problem according to some number of parameters, not calculating the grids but applying one of well-known methods of investigation of the extremum by many variables.

In similar cases, for the sake of saving time, it is recommend to derive for printing not all the obtained trajectories but only the final results of the calculations and only for several variants is it possible to derive trajectories for the use of them in calculations of loads, stability of motion and controllability, in the carrying out of thermal and aerodynamic designs, and so forth. As a rule, for purposes of selection of design parameters there is used a system of equations of motion (14.25) written in the assumption of an ideal control system ($a_0 = \infty$), but considering the presence of angles of attack. The program of pitch angle for a single-stage rocket is given in the form of a one-parameter or two-parameter family of curves, where for one parameter there is taken the maximum value of the angle of attack on the subsonic section of the trajectory and for the second, the origin of turn of the axis of the rocket in pitch.[1]

For a two-stage rocket there is the possibility of the variation of two more parameters of the program: the initial value of the pitch angle on the second stage and angular velocity taken as constant for the given trajectory.

Absolutely analogous calculations are conducted for exposure of the influence of deviations in comparatively small limits of basic design parameters on the flying range. The obtained change in distance attributed to the increase in the investigated parameter, is equated by a corresponding derivative if one were to solve the problem in a linear formulation.

It is necessary only to note that in the carrying out of similar calculations the program of pitch angle should not be taken the same for perturbed trajectories as for undistrubed, but each time should be selected from the guaranteed condition of the maximum of range. Derivatives obtained as a result of such calculations can be used in certain other design problems. Let us assume that it is required, for example, to establish between some two design parameters λ_1 and λ_2 a relation corresponding to the constancy of the maximum range of the flight. The relation of derivatives

$$-\frac{\partial L}{\partial \lambda_2} \Big/ \frac{\partial L}{\partial \lambda_1} = \frac{d\lambda_1}{d\lambda_2}$$

---

[1] For the assignment of the program of the pitch angle see in greater detail in Part Four.

gives to us the needed value without carrying out additional calculations. Usually with-design calculations we are interested in mean values of flying characteristics of the rocket, and therefore the influence of the rotation of earth will be disregarded. However in the programming of problems for a machine reading, in equations of motion it is recommend to preserve the appropriate members, since in a number of cases the investigations taking into account the rotation of earth are needed very much. In remaining cases superfluous operations can be avoided, setting in initial data the angular velocity of the rotation of earth equal to zero.

In many cases the determination of the full flying range is expedient conducted not with the help of transition to formulas of the elliptic theory, but by continuing integration of equations of motion up to the moment of the encounter with earth. It is more convenient for calculation of the powered and free-flight section of the flight to use the same system of equations of motion, excluding during calculation of the free-flight section of trajectory members connected with the operation of the engine and control system (setting, for example, the thrust to equal to zero).

With a machine reading the most convenient is the method of Runge-Kutta integration, where the step of integration should be slected automatically, proceeding from the assigned accuracy of the calculation. This guarantees both against the uneconomical expenditure of machine time (if the step is assigned very small) and insufficient accuracy of the calculation. In other respects it is necessary to hold to recommendations general for calculations of trajectories with any accepted system of equations of motion and given in Chapter VII and also in Chapter XI, with respect to the selection of the program of the pitch angle.

§ 26. Determination of the Speed of Encounter of the Rocket with a Target

It is frequently important to determine the speed of encounter of a rocket with a target, since with this speed are connected conditions of motion of the rocket before encounter. This can be done by the following method belonging to Prof. V. P. Vetchinkin [1].

Let us assume that with the entrance into the atmosphere on a descending phase of the trajectory the rocket has the following initial parameters of motion: altitude h_H, speed v_H and angle of inclination of the velocity vector to the local horizon ϑ_H[1]. Let us assume the trajectory of the rocket on the atmospheric section to be rectilinear. This will not introduce great error into the calculation, since the true trajectory is insignificantly deviated from the rectilinear. Further we will replace the drag coefficient c_x by its mean value and disregard the dependence g on altitude. Thus, we make the assumptions

$$\vartheta = \vartheta_н = \text{const},$$
$$c_x = c_{x\,ср} = \text{const},$$
$$g = g_0 = \text{const}.$$

The equation of motion on the atmospheric section will have the form

$$\frac{dv}{dt} = -\frac{X}{m} + g\sin\vartheta = -\frac{c_{x\,ср}\rho_0 S}{2m}\frac{\rho}{\rho_0}v^2 + g_0\sin\vartheta_н, \qquad (26.1)$$

where m — the mass of the rocket; S — area of midsection; ρ — air density at an altitude; ρ_0 — air density on earth.

Equation of motion can be written thus:

$$\frac{dv}{dt} = \frac{dv}{dh}\frac{dh}{dt} = -\frac{c_{x\,ср}\rho_0 S}{2m}\frac{\rho}{\rho_0}v^2 + g_0\sin\vartheta_н; \qquad (26.2)$$

---

[1] In this paragraph, for convenience of calculations the positive direction of the reading of angle ϑ is accepted from the horizon clockwise.

here h is the altitude of the rocket above the surface of earth; ρ/ρ_0 is the relative air density.

It is obvious that

$$\frac{dh}{dt} = -v\sin\theta_H. \tag{26.3}$$

Inserting (26.3) in (26.2), we obtain

$$v\sin\theta_H\frac{dv}{dh} = \frac{c_{x\,cp}\rho_0 S}{2m}\frac{\rho}{\rho_0}v^2 - g_0\sin\theta_H,$$

or

$$\frac{d(v^2)}{dh} = \frac{c_{x\,cp}\rho_0 S}{m\sin\theta_H}\frac{\rho}{\rho_0}v^2 - 2g_0. \tag{26.4}$$

We designate

$$k = \frac{c_{x\,cp}\rho_0}{2\sin\theta_H}\frac{S}{m} = \frac{g\rho_0}{2\sin\theta_H}\frac{c_{x\,cp}}{p_H},$$

where

$$p_H = \frac{G}{S}\left[\frac{кг}{м^2}\right].$$

Then equation (26.4) will take the form

$$\frac{d(v^2)}{dh} = 2k\frac{\rho}{\rho_0}v^2 - 2g_0.$$

Integrating this equation from h = h_H up to h = 0, we find the final expression for the speed of encounter with a target

$$v_C^2 = v_H^2 e^{-2k\int_0^{h_H}\frac{\rho}{\rho_0}dh} + \int_0^{h_H} 2g_0 e^{-2k\int_0^{h}\frac{\rho}{\rho_0}dh}\,dh. \tag{26.5}$$

Considering that with the made assumptions

$$\int_0^h \frac{\rho}{\rho_0}dh = -\int_0^h \frac{-g\rho\,dh}{g\rho_0} = -\frac{1}{g\rho_0}\int_{p_0}^{p} dp = \frac{p_0 - p}{g\rho_0},$$

formula (26.5) can be written in the following way:

$$v_C^2 = v_H^2 e^{\frac{-c_{x\,cp}(p_0 - p_H)}{p_H\sin\theta_H}} + 2g\int_0^{h_H} e^{-\frac{c_{x\,cp}(p_0 - p)}{p_H\sin\theta_H}}\,dh. \tag{26.6}$$

By formula (26.6) calculations were made graphs were plotted (Fig. 26.1-26.3 and Fig. VI-VIII of the Appendex). With this for the initial altitude of the rectilinear section that altitude was taken on which the assigned initial speed v_H the acceleration from drag consists of 1/10 of the acceleration imparted to the rocket by gravity, i.e., equal to

$$0{,}1g\sin\theta_H \approx \sin\theta_H.$$

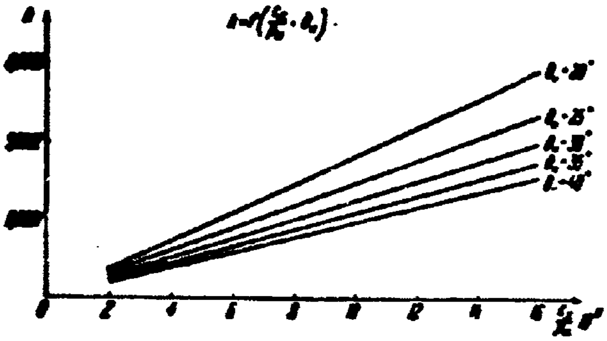

Fig. 26.1

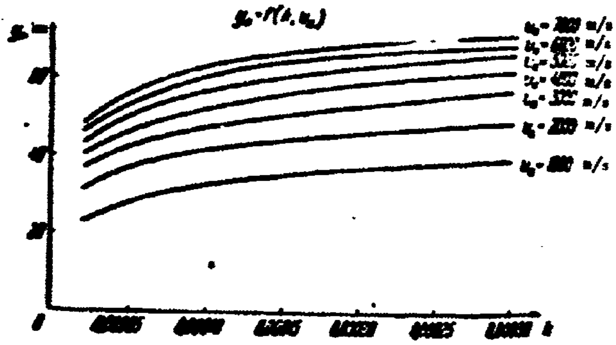

Fig. 26.2

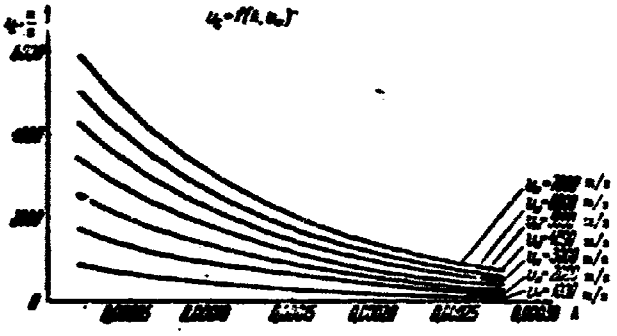

Fig. 26.3

Thus altitude is found from equality

$$\frac{1}{2m} c_{x\,cp} \rho_0 \frac{\rho}{\rho_0} v_n^2 S = \sin\theta_n,$$

consequently,

$$\frac{\rho}{\rho_0} = \frac{2G\sin\theta_n}{g\rho_0 c_{x\,cp} S v_n^2} = \frac{1}{k v_n^2}. \tag{26.7}$$

According to equation (26.7) the graph $y_H = f(k, v_H)$ is plotted (Fig. 26.2).

The calculation of speed v_C by the given graphs is produced in the following way. Knowing for the given rocket c_{xcp} and p_M, and also the angles ϑ_H, equal to the angle at the end of the powered section, from the graph of Fig. 26.1 (Fig. VI we find k. From the calculation of the powered section there should be known v_K and y_K. From the graph of Fig. 26.2 (Fig. VII) with respect to v_K and k we find y_H; it is most probable that the value of y_H will equal y_K. Then we determine the value of speed by the formula

$$v_H = v_K + \frac{g(y_K - y_H)}{v_K}. \qquad (26.8)$$

With respect to the new value of speed v_H and value k from the graph of Fig. 26.3 (Fig. VIII) we find the speed of encounter v_C.

Calculations made from the graphs show that the error in determining v_C, as compared to numerical integration, is quite permissible for design calculations.

C H A P T E R VII

BALLISTIC CHECK CALCULATIONS

Afterwards, with the help of the above-stated method of design calculation main design parameters of rocket are selected: $\mu_{\text{к}} = \frac{G_{\text{к}}}{G_0}$ — ratio of the final weight to the initial; $P_{\text{уд.п}}$ — specific thrust in a vacuum; $\nu_0 = \frac{G_0}{P_0}$ — launch ratio of weight to thrust; $P_{\text{м}} = \frac{G_0}{S}$ — load on midsection; $\Delta P_{\text{уд}} = P_{\text{уд.п}} - P_{\text{уд0}}$ — the altitude of performance of the nozzle expressed as the difference between specific thrusts in a vacuum and on Earth; $c_x(M)$ — law of drag. It is possible to approach a more full and detailed designing with a further more precise definition of the enumerated parameters.

This more exact data should correspond to more exact calculations, which are conducted by equations of motion obtained in one part. The most suitable here are systems of equations (14.30) and (14.25) for the calculation of the powered-flight trajectory and systems of equations (15.19) and (15.12) for the calculation of the free-flight section.

Simultaneously with these calculations there is conducted a selection the "program" or form of trajectory about which should be carried out motion on the powered section and in accordance with which instruments giving the rocket this motion are designed. Questions connected with the selection of the program will be examined in the fourth part of the book.

After a more precise definition of main design parameters of the rocket and the selection of a program checking calculations are produced which consider certain peculiarities of the control system, a fuller scheme of action of the forces and, if it is necessary, a concrete point of launch and direction of firing. The most suitable for this purpose are systems of equations (14.20)-(14.23) and (16.18) for the calculation of the powered-flight trajectory and the system of equations (15.3)-(15.7) for the calculation of the section of free flight. On the basis of these calculations there are compiled preliminary tables of firing and conducted flight tests of the designed rocket.

During calculation of the trajectory it is necessary first of all to select a certain system of differential equations, proceeding from the required accuracy of determination of the flying range and other elements of the trajectory interesting to us and in accordance with the presence and accuracy of initial data necessary for carrying out the calculation. Appropriate recommendations on application of a certain system of equations of motion were given with the derivation of these equations.

In this chapter we will examine concrete systems of the most commonly used equations for calculation and will give recommendations for carrying out numerical calculations.

§ 27. Calculation of Powered-Flight Trajectory

One of the most complete for calculation of the powered-flight trajectory is the system of equations (16.18):

$$\left.\begin{aligned}
\frac{dv}{dt} &= \frac{1}{m}(P - X_{1p} - c_x qS) - g\sin\theta - \frac{x}{r} g\cos\theta.\\
\frac{d\theta}{dt} &= \frac{1}{v}\left\{\frac{\alpha}{m}\left[P - X_{1p} + \frac{l_1 - x_2}{l_1 - x_T} c'_y qS\right] - g\cos\theta +\right.\\
&\qquad \left. + \frac{x}{r} g\sin\theta\right\} + 2\omega_3 \cos\varphi_r \sin\psi.\\
\frac{d\sigma}{dt} &= \frac{1}{v}\left\{\frac{\beta}{m}\left[P - X_{1p} + \frac{l_1 - x_2}{l_1 - x_T} c'_y qS\right] + \sigma\dot{g}\sin\theta\right\} -\\
&\qquad - 2\omega_3(\sin\varphi_r \cos\theta - \cos\varphi_r \cos\psi \sin\theta).\\
\frac{dx}{dt} &= v\cos\theta.\\
\frac{dy}{dt} &= v\sin\theta.\\
\frac{dz}{dt} &= -v\sigma.
\end{aligned}\right| \qquad (27.1)$$

Here

$$m = m_0 - \int_0^t \dot{m}\,dt,$$

where m_0 – mass of the rocket at the time of breakaway from the launching pad.

The value m_0 does not coincide with the mass of a completely filled rocket, but is less than it by a magnitude of the so-called pre-launch flow rate, by which is understood the mass of fuel expended prior to the moment of breakaway of the rocket from the launching pad. This moment is characterized by an equality of thrust and weight of the rocket, and the corresponding quantity of fuel expended up to this moment is determined on the basis of available statistics from results of bench tests of the engine. Thus for the zero of time in ballistic calculations is taken the moment of breakaway of the rocket from the launcher.

The flow rate per second of mass $\dot{m}$ is determined by characteristics of the propulsion system. During calculation should be considered change $\dot{m}$ depending upon ballistic parameters of motion and changes of conditions of the fuel feed dependent on them for every component separately. Calculation of the character of accretion $\dot{m}$ after the switching of the engine and fall $\dot{m}$ after the turning off of the engine is obligatory.

Thrust is determined by the formula

$$P = \frac{\dot{m}}{\dot{m}_0}(P_0 + S_a p_0) - S_a p = c_1 \frac{\dot{m}}{\dot{m}_0} - c_2 \frac{p}{p_0}, \qquad (27.2)$$

where for quantities not variable during flight there are introduced designations

$$c_1 = P_0 + S_a p_0,$$
$$c_2 = S_a p_0;$$

P_0 and $\dot{m}_0$ are nominal values of thrust and flow rate per second of mass on earth. Formula (27.2) considers throttle (for small changes of flow rate) and altitude characteristics of the engine. Ratio p/p_0 is rate taken from tables of standard (normal) atmosphere depending upon the altitude of flight.

The drag of the jet vanes is equal to

$$X_{lp} = 4Q_{p0} + \lambda(\delta_1^2 + 2\delta_2^2 + \delta_3^2) + 4\lambda \frac{a^2}{2} =$$
$$= 4\left(Q_{p0} + \lambda \frac{a^2}{2}\right) + \lambda(\delta_1^2 + 2\delta_2^2 + \delta_3^2).$$

where Q_{p0} and λ are determined experimentally; a is the average amplitude of oscillations of control surfaces around the program position.

Since in the calculation angles of deviation of control surfaces 1 and 3 do not appear, then for a determination of the total loss of thrust on the control surfaces this formula is used

$$X_{lp} = 4\left(Q_{p0} + \lambda \frac{a^2}{2}\right) + 2\lambda\delta_2^2 = c_3 + c_4\delta_2^2,$$

where there is designated

$$c_3 = 4\left(Q_{p0} + \lambda \frac{a^2}{2}\right),$$
$$c_4 = 2\lambda.$$

Member $c_x qS$, for convenience of calculation, is presented in a somewhat different form:

$$c_x qS = c_x \frac{\rho}{\rho_0} v^2 \frac{S\rho_0}{2} = c_5 \frac{\rho}{\rho_0} v^2 c_x,$$

where there is designated

$$c_5 = \frac{S\rho_0}{2};$$

c_x is taken depending upon the Mach number M (or quantity $w = Ma_0$, where a_0 is the speed of propagation of sound on earth) and the Reynolds number (or altitude of the flight) with a correction for angle of attack; the ratio ρ/ρ_0 is taken from tables of standard atmosphere depending upon the altitude of the flight.

The acceleration of gravity g is calculated by the formula

$$g = g_0 \frac{R^2}{r^2},$$

where the acceleration of gravity at the surface of Earth g_0 should be determined depending upon the latitude of the point of launch φ_r by the formula

$$g_0 = 9.7805 + 0.0519 \sin^2 \varphi_r \left[-, \right].$$

The altitude of the flight of the rocket above the surface of Earth is determined by the formula

$$h = y + \Delta h,$$

where

$$\Delta h = \frac{z^2}{2R}.$$

If the distance of the powered section is too great, then the altitude should be calculated thus:

$$h = \sqrt{(R+y)^2 + x^2 + z^2} - R,$$

where

$$R = 6371 \text{ км}.$$

It is convenient to record member $\frac{l_1 - x_{д}}{l_1 - x_{т}} c'_y qS$ in a somewhat different form:

$$\frac{l_1 - x_{д}}{l_1 - x_{т}} c'_y qS = c_{\delta} \frac{\rho}{\rho_0} v^2 \frac{\frac{l_1}{l} - c_{д}}{\frac{l_1}{l} - \frac{x_{т}}{l}} c'_y. \qquad (27.2)$$

Here $c'_y = \frac{dc_y}{d\alpha}$ and $c_{Д}$ are taken from tables of aerodynamic coefficients[1] depending upon the M number; l_1/l is the quantity constant for the rocket; $x_{т}/l$ is taken from the graph or table[1] which is most convenient of all to have depending upon the mass of the rocket.

For the integration of equations of motion (27.1) there is used the method of Adams. For the calculation of initial points there is applied the method of consecutive approaches, proposed by Acad. A. N. Krylov, who recommended the Adams method for solution of the basic problem of external ballistics. During integration of equations of motion (27.1) it is necessary to use certain final relations:

$$\beta = -A(\sigma - \gamma_1 \sin\theta + \gamma_2 \cos\theta),$$
$$\alpha = A(\varphi_{пр} - \gamma_3 - \theta),$$
$$\Delta\varphi = -(1 - A)(\varphi_{пр} - \gamma_3 - \theta),$$

where A designates the quantity

$$A = \frac{2a_0 R'(l_1 - x_{т})}{2a_0 R'(l_1 - x_{т}) + c'_y qS(x_{д} - x_{т})} = \frac{2a_0 R'\left(\frac{l_1}{l} - \frac{x_{т}}{l}\right)}{2a_0 R'\left(\frac{l_1}{l} - \frac{x_{т}}{l}\right) + c'_y qS\left(c_{д} - \frac{x_{т}}{l}\right)}.$$

Here a_0 is the coefficient of the static dependence between the angle of deviation of the rocket's axis and angle of deviation of the jet vanes 2 and 4:

$$\delta_2 = \delta_4 = a_0 \Delta\varphi.$$

---

[1] These are formulated in the process of designing of the rocket in the form of weight, centering and aerodynamic designs.

Value a_0 depends on characteristics of control system and controls and should be assigned. Equations of this paragraph allow both a constant and variable value of a_0. Further, R' is a derivative of lift of the jet vane according to the angle of its deviation. R' depends on the configuration and area of the jet vane and characteristics of the gas stream. Considering the angles of deviation of the control surfaces small, and the characteristics of the stream and control surfaces themselves constant in flight, it is possible to take R' as some mean value. In reality, because of the burning of jet vanes and the change in characteristics of the stream, R' does not remain constant, but this change, as experiments show, is small.

The values γ_1, γ_2, γ_3 necessary for calculations, considering the rotation of the launch system of coordinates, are calculated by the formulas (16.4):

$$\left.\begin{aligned} \gamma_1 &= \omega_3 t \cos\varphi_r \cos\psi. \\ \gamma_2 &= \omega_3 t \sin\varphi_r. \\ \gamma_3 &= -\omega_3 t \cos\varphi_r \sin\psi. \end{aligned}\right\} \qquad (27.4)$$

where

$$\omega_3 = 7.2921 \cdot 10^{-5}\left[\frac{1}{s}\right];$$

φ_r – latitude of the point of launch; ψ – azimuth of the direction of firing.

The true position of the axis of the rocket is determined by angles

$$\varphi = \varphi_{\text{пр}} + \Delta\varphi.$$

$$\xi = (1 - A)(\sigma - \gamma_1 \sin\theta + \gamma_2 \cos\theta).$$

The third and sixth equation of the system (27.1) can be integrated from a certain moment t ≠ 0. As such moment it is recommended to select the end of the vertical section with initial conditions

$$\theta = 90^\circ - \gamma_3.$$

$$\sigma = \gamma_1.$$

In all calculations the program change of the angle of inclination of the axis of the rocket φ_{np} should be assigned in the form of tables with an interval equal to the step of numerical integration.

Equations of motion obtained in § 14 differ from equations of § 16 only by the absence of members considering the departure of gyroscopes with respect to the terrestrial system of coordinates due to the rotation of the latter. Therefore, all calculations are somewhat simplified. The system of equations itself has such a form:

$$
\left.
\begin{aligned}
&\frac{dv}{dt}=\frac{1}{m}(P-X_{1p}-c_xqS)-g\sin\theta-x\frac{g}{r}\cos\theta,\\
&\frac{d\theta}{dt}=\frac{1}{v}\left\{\frac{\alpha}{m}\left[P-X_{1p}+\frac{\frac{l_1}{l}-c_z}{\frac{l_1}{l}-\frac{x_T}{l}}c_y'qS\right]-\right.\\
&\qquad\left.-g\cos\theta+x\frac{g}{r}\sin\theta\right\}+2\omega_3\cos\varphi_r\sin\psi.\\
&\frac{d\sigma}{dt}=\frac{1}{v}\left\{\frac{\beta}{m}\left[P-X_{1p}+\frac{\frac{l_1}{l}-c_z}{\frac{l_1}{l}-\frac{x_T}{l}}c_y'qS\right]+\sigma g\sin\theta\right\}-\\
&\qquad-2\omega_3(\sin\varphi_r\cos\theta-\cos\varphi_r\cos\psi\sin\theta).\\
&\frac{dx}{dt}=v\cos\theta,\\
&\frac{dy}{dt}=v\sin\theta,\\
&\frac{dz}{dt}=-v\sigma.
\end{aligned}
\right\}
\qquad (27.5)
$$

where

$$
\begin{aligned}
&\alpha=A(\varphi_{\text{пр}}-\theta),\quad \beta=-A\sigma,\\
&A=\frac{2a_\varphi R'\left(\frac{l_1}{l}-\frac{x_T}{l}\right)}{2a_\varphi R'\left(\frac{l_1}{l}-\frac{x_T}{l}\right)+c_y'qS\left(c_z-\frac{x_T}{l}\right)}.\\
&\delta_3=a_\varphi\Delta\varphi.\\
&\Delta\varphi=-(1-A)(\varphi_{\text{пр}}-\theta).\\
&\varphi=\varphi_{\text{пр}}+\Delta\varphi.
\end{aligned}
$$

Everything said with respect to the calculation of separate values for the system of equations (27.1) and its integration remains in force for the system (27.5).

For the calculation of the powered-flight trajectory, neglecting the rotation of earth and in the assumption of an "ideal" control system ($a_\varphi = \infty$), it is necessary to use the system of equations (14.25):

$$
\begin{aligned}
&\frac{dv}{dt}=\frac{1}{m}(P-X_{1p}-c_xqS)-g\sin\theta-\frac{x}{r}g\cos\theta,\\
&\frac{d\theta}{dt}=\frac{1}{v}\left\{\frac{\varphi_{\text{пр}}-\theta}{m}\left[P-X_{1p}+\frac{\frac{l_1}{l}-c_z}{\frac{l_1}{l}-\frac{x_T}{l}}c_y'qS\right]-\right.\\
&\qquad\qquad\left.-g\cos\theta+\frac{x}{r}g\sin\theta\right\}.\\
&\frac{dx}{dt}=v\cos\theta,\\
&\frac{dy}{dt}=v\sin\theta,\\
&\alpha=\varphi_{\text{пр}}-\theta.
\end{aligned}
$$

If a change in θ in time is known beforehand, and the program change of the direction of the axis is unknown, then the second equation of the system written above can be used for the determination of α and $\varphi_{\text{пр}}$:

$$\alpha = \frac{m\left(v\frac{d\theta}{dt} + g\cos\theta - \frac{x}{r}g\sin\theta\right)}{P - X_{1\text{п}} + \frac{l_1 - x_2}{l_1 - x_1}c_y' qS},$$

$$\varphi_{\text{пр}} = \theta + \alpha.$$

Simplifications, as a result of which the last system was obtained permit in the calculations disregarding the change in flow rate per second of mass and the change in losses of thrust on the jet vanes depending upon their deflection and taking as their mean value ($X_{1p} = X_{1p.cp}$ is the average loss of thrust on the jet vanes). In exactly the same way the thrust itself is determined by neglecting these changes, taking into account only the altitude performance of the engine

$$P = P_0 + S_a p_0\left(1 - \frac{p}{p_0}\right).$$

With the necessity of determining the angle of deviation of program control surfaces it is possible to use the expression

$$\delta_2 = -\frac{c_y' qS(x_2 - x_1)}{2R'(l_1 - x_1)}(\varphi_{\text{пр}} - \theta).$$

Finally, the simplest system of equations of motion is the system (14.30):

$$\frac{dv}{dt} = \frac{1}{m}(P - X_{1\text{п}} - c_x qS) - g\sin\varphi_{\text{пр}},$$

$$\frac{dx}{dt} = v\cos\varphi_{\text{пр}},$$

$$\frac{dy}{dt} = v\sin\varphi_{\text{пр}},$$

obtained from the system of equations (14.25), if we disregard the angle of attack α and member $\frac{x}{r}g$.

Since the system of equations (14.30) is used for rough calculations necessary mainly in the designing of the rocket, many parameters and initial data can be not completely accurate. Therefore, here there is no sense in considering the accretion of thrust after switching on the engine. Flow rate per second and loss of thrust on jet vanes are taken on the entire powered section to be constant. A change in c_x depending upon altitude is also possible not to consider. Owing to these peculiarities the step of integration can be selected sufficiently large, i.e., up to 4-5 s, and sometimes more.

§ 28. Calculation of the Coasting Trajectory

In § 15 it was shown that the main factors affecting coincidence of calculation range with the actual are perfection of the control system and accuracy of calculation of the section of free flight. During ideal operation of the range control system any error in the calculation of free flight section will lead to a divergence between the calculation of sections of free flight with firing at the outlined target or with compilation of tables of firing is absolutely necessary. Such accuracy is satisfied by the system of equations (16.22):

$$\left.\begin{aligned}
\frac{dv_x}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_x-\frac{g_r'}{r}(x-x_c)-\frac{g_\omega}{\omega_3}\omega_{3x}+\\
&\quad+a_{11}(x-x_c)+a_{12}(y-y_c)+\\
&\qquad+a_{13}(z-z_c)+b_{12}v_y+b_{13}v_z,\\
\frac{dv_y}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_y-\frac{g_r'}{r}(y-y_c)-\frac{g_\omega}{\omega_3}\omega_{3y}+\\
&\quad+a_{21}(x-x_c)+a_{22}(y-y_c)+\\
&\qquad+a_{23}(z-z_c)+b_{21}v_x+b_{23}v_z,\\
\frac{dv_z}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_z-\frac{g_r'}{r}(z-z_c)-\frac{g_\omega}{\omega_3}\omega_{3z}+\\
&\quad+a_{31}(x-x_c)+a_{32}(y-y_c)+\\
&\qquad+a_{33}(z-z_c)+b_{31}v_x+b_{32}v_y,\\
\frac{dx}{dt} &= v_x,\\
\frac{dy}{dt} &= v_y,\\
\frac{dz}{dt} &= v_z,
\end{aligned}\right| \qquad (2[illegible].1)$$

where

$$\begin{aligned}
k &= \frac{S\rho_0}{2m};\\
x_c &= a\alpha\sin 2\varphi_r\cos\psi,\\
y_c &= -a(1-\alpha\sin^2\varphi_r),\\
z_c &= -a\alpha\sin 2\varphi_r\sin\psi,\\
a_{11} &= \omega_3^2(\sin^2\varphi_r+\cos^2\varphi_r\sin^2\psi),\\
a_{21} &= a_{12} = -\omega_3^2\sin\varphi_r\cos\varphi_r\cos\psi,\\
a_{13} &= a_{31} = \omega_3^2\cos^2\varphi_r\sin\psi\cos\psi,\\
a_{22} &= \omega_3^2\cos^2\varphi_r,\\
a_{23} &= a_{32} = \omega_3^2\sin\varphi_r\cos\varphi_r\sin\psi,\\
a_{33} &= \omega_3^2(\sin^2\varphi_r+\cos^2\varphi_r\cos^2\psi),\\
b_{12} &= -b_{21} = -2\omega_3\cos\varphi_r\sin\psi,\\
b_{13} &= -b_{31} = -2\omega_3\sin\varphi_r,\\
b_{23} &= -b_{32} = 2\omega_3\cos\varphi_r\cos\psi.
\end{aligned}$$

All these values for the given trajectory are constant and their values are calculated beforehand; c_x is determined from tables of aerodynamic coefficients depending upon the M number (or value w) and altitude. Dependence c_x on the angle of attack is not considered, since it is assumed the flight occurs without an angle of attack.

Values $\frac{\rho}{\rho_0}$ and $\sqrt{\frac{T_0}{T}}$ are determined from tables of standard atmosphere depending upon the altitude.

Components of the acceleration of gravity g_r' and g_ω are calculated by the formulas

$$\begin{aligned}
g_r' &= \frac{fM}{r^2}-\frac{\varepsilon}{r^4}(5\sin^2\varphi_ш-1),\\
g_\omega &= \frac{2\varepsilon}{r^4}\sin\varphi_ш.
\end{aligned}$$

For the determination of r and $\varphi_{\text{Ц}}$ these formulas are used

$$r = \sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2}.$$

$$\sin\varphi_{\text{ц}} = \frac{(x - x_c)\cos\varphi_c\cos\psi + (y - y_c)\sin\varphi_c - (z - z_c)\cos\varphi_c\sin\psi}{r}.$$

Altitude h can be defined by formula

$$h = r - a(1 - \alpha\sin^2\varphi_{\text{ц}}).$$

For the calculation of angles determining the direction of the tangent to the trajectory and the magnitude of velocity v we use the following relations:

$$\operatorname{tg}\theta = \frac{v_y}{v_x}.$$

$$\operatorname{tg}\sigma = -\frac{v_z\cos\theta}{v_x}.$$

$$v = \frac{v_x}{\cos\theta\cos\sigma}.$$

Constants entering into the equation of motion have the values

$$a = 6\,378\,245 \text{ м}.$$

$$\alpha = \frac{1}{298{,}3}.$$

$$fM = 3{,}9862 \cdot 10^{14}\,\frac{m^3}{s^2}.$$

$$\mu = 26{,}245 \cdot 10^{24}\,\frac{m^5}{s^2}.$$

Calculation is produced by-means of numerical integration of equations of motion by the Adams method. Initial conditions for integration are parameters of the end of the powered-flight trajectory.

The examined system of equations of motion for free-flight section (28.1) is not obligatory for all ranges. For ranges not exceeding 500 km it is possible to use equations given in § 15 in which the flatness of earth is not considered:

$$\frac{dv_x}{dt} = -kc_x\frac{\rho}{\rho_0}vv_x - \frac{g}{r}x + a_{11}x + a_{12}(R + y) + \\ + a_{13}z + b_{12}v_y + b_{13}v_z.$$

$$\frac{dv_y}{dt} = -kc_x\frac{\rho}{\rho_0}vv_y - \frac{g}{r}(R + y) + a_{21}x + \\ + a_{22}(R + y) + a_{23}z + b_{21}v_x + b_{23}v_z.$$

$$\frac{dv_z}{dt} = -kc_x\frac{\rho}{\rho_0}vv_z - \frac{g}{r}z + a_{31}x + \\ + a_{32}(R + y) + a_{33}z + b_{31}v_x + b_{32}v_y.$$

The remaining equations and relations remain as before, with the exception of expressions for g_r', g_ω, r and h, instead of which one should use formulas

$$g = \frac{fM}{r^2}.$$

$$r = \sqrt{(R + y)^2 + x^2 + z^2}.$$

$$h = r - R.$$

If the flatness of earth is not considered, then it is recommended to use the calculations only as long as drag has an influence on the motion. The subsequent part of the free-flight section should be calculated by formulas of elliptic theory obtained in Chapter V. With this it is necessary to use the transition from the absolute motion to the relative, as was shown in § 17 and § 20. The sequence of calculations is the following.

Having initial data x_H, y_H, z_H, $\dot{x}_H$, $\dot{y}_H$, $\dot{z}_H$, we turn to the auxiliary system of coordinates:

$$x'_{н} = -(x_{н}\cos\psi - z_{н}\sin\psi)\sin\varphi_r + (R + y_{н})\cos\varphi_r.$$
$$y'_{н} = x_{н}\sin\psi + z_{н}\cos\psi.$$
$$z'_{н} = (x_{н}\cos\psi - z_{н}\sin\psi)\cos\varphi_r + (R + y_{н})\sin\varphi_r.$$

We determine the components of absolute velcoity in this auxiliary system:

$$\dot{x}'_{н} = -(\dot{x}_{н}\cos\psi - \dot{z}_{н}\sin\psi)\sin\varphi_r + \dot{y}_{н}\cos\varphi_r - \omega_3 y'_{н}.$$
$$\dot{y}'_{н} = \dot{x}_{н}\sin\psi + \dot{z}_{н}\cos\psi + \omega_3 x'_{н}.$$
$$\dot{z}'_{н} = (\dot{x}_{н}\cos\psi - \dot{z}_{н}\sin\psi)\cos\varphi_r + \dot{y}_{н}\sin\varphi_r.$$

We find spheric coordinates of the initial point (longitude is read off from the point of launch)

$$\operatorname{tg}\lambda_{н} = \frac{y'_{н}}{x'_{н}}, \quad \operatorname{tg}\varphi_{rн} = \frac{z'_{н}\cos\lambda_{н}}{x'_{н}}, \quad r_{н} = \frac{z'_{н}}{\sin\varphi_{rн}}.$$

We determine the components of absolute velocity about the meridian, parallel and radius of Earth:

$$v'_{\varphi н} = -(\dot{x}'_{н}\cos\lambda_{н} + \dot{y}'_{н}\sin\lambda_{н})\sin\varphi_{rн} + \dot{z}'_{н}\cos\varphi_{rн}.$$
$$v'_{\lambda н} = -\dot{x}'_{н}\sin\lambda_{н} + \dot{y}'_{н}\cos\lambda_{н}.$$
$$v'_{rн} = (\dot{x}'_{н}\cos\lambda_{н} + \dot{y}'_{н}\sin\lambda_{н})\cos\varphi_{rн} + \dot{z}'_{н}\sin\varphi_{rн}.$$

We calculate the absolute azimuth

$$\operatorname{tg}\psi'_{н} = \frac{v'_{\lambda н}}{v'_{\varphi н}},$$

angle of inclination of the tangent to the absolute trajectory

$$\operatorname{tg}\vartheta'_{н} = \frac{v'_{rн}\cos\psi'_{н}}{v'_{\varphi н}} = \frac{v'_{rн}\sin\psi'_{н}}{v'_{\lambda н}}$$

and quantity of absolute velocity

$$v'_{н} = \frac{v'_{rн}}{\sin\vartheta'_{н}}.$$

We further determine the auxiliary parameter ν'_H:

$$\nu'_{н} = \frac{v'^2_{н} r_{н}}{fM}\left(fM = 3.9862\cdot10^{14}\ \frac{m^3}{s^2}\right).$$

we calculate the central angle in absolute motion

$$\operatorname{tg}\frac{\beta'}{2} = \frac{\nu'_{н}\operatorname{tg}\vartheta'_{н}}{1 + \operatorname{tg}^2\vartheta'_{н} - \nu'_{н}}.$$

calculate the auxiliary quantity x'_H from the relation

$$\cos x'_{н} = \frac{1 - \nu'_{н}}{\sqrt{1 - (2 - \nu'_{н})\nu'_{н}\cos^2\vartheta'_{н}}}.$$

where $0 < x'_{н} < \pi$, and find the time of flight by the formula

$$t' = \frac{2r_{н}v'_{н}}{v'_{н}(2-\nu'_{н})}\left(\sin\theta'_{н} + \frac{x'_{н}}{\sqrt{(2-\nu'_{н})\nu'_{н}}}\right).$$

We find the geographic coordinates of the end point (lying at one altitude with the initial point $r_p = r_H$)

$$\sin\varphi_{rp} = \sin\varphi_{rн}\cos\beta' + \cos\varphi_{rн}\sin\beta'\cos\psi'_{н},$$

$$\sin(\lambda_p - \lambda_{н} + \omega_з t') = \frac{\sin\beta'\sin\psi'_{н}}{\cos\varphi_{rp}}$$

and determine the azimuth at the end point

$$\sin\psi'_p = \frac{\sin\psi'_{н}\cos\varphi_{rн}}{\cos\varphi_{rp}}.$$

Then obtain the components speed at the end point:

$$v_{\varphi p} = v'_{н}\cos\theta'_{н}\cos\psi'_p,$$

$$v_{\lambda p} = v'_{н}\cos\theta'_{н}\sin\psi'_p - \omega_з r_{н}\cos\varphi_{rp},$$

$$v_{rp} = -v'_{н}\sin\theta'_{н} = -v'_{rн},$$

and find the azimuth and angle of inclination of the tangent at the end point in relative motion

$$\operatorname{tg}\psi_p = \frac{v_{\lambda p}}{v_{\varphi p}},$$

$$\operatorname{tg}\theta_p = \frac{v_{rp}\cos\psi_p}{v_{\varphi p}} = \frac{v_{rp}\sin\psi_p}{v_{\lambda p}},$$

also magnitude of relative speed

$$v_p = \frac{v_{rp}}{\sin\theta_p}.$$

To determine only the full range the formulas of elliptic theory can be used up to the point of impact on earth, since atmospheric drag on the descending phase of the trajectory for distances over 600-800 km does not have considerable influence on the range.

With the necessity, to determine, besides range, other elements of the trajectory on the fall section in the atmosphere, for example, acceleration, speed. etc., it follows, starting from moment t', again to use the system of equations (28.1).

In this case after calculation of the section of fall in the atmosphere with initial data v_p, θ_p, φ_{rp}, ψ_p, $h_p = h_{н}$ by equations (28.1), we again introduce an auxiliary system of coordinates in which we determine coordinates of the point of impact by the formulas

$$x^c_C = -(x_C\cos\psi_p - z_C\sin\psi_p)\sin\varphi_{rp} + (R + y_C)\cos\varphi_{rp},$$

$$y^c_C = x_C\sin\psi_p + z_C\cos\psi_p,$$

$$z^c_C = (x_C\cos\psi_p - z_C\sin\psi_p)\cos\varphi_{rp} + (R + y_C)\sin\varphi_{rp}.$$

Then we find geographic coordinates of the point of impact

$$\operatorname{tg}(\lambda_C - \lambda_p) = \frac{y_C^{*}}{x_C^{*}},$$

$$\operatorname{tg}\varphi_{rC} = \frac{z_C^{*}\cos(\lambda_C - \lambda_p)}{x_C^{*}},$$

after which there is determined the azimuth of direction on the point of impact

$$\operatorname{ctg}\psi_{0C} = \frac{\operatorname{tg}\varphi_{rC}\cos\varphi_r - \cos\lambda_C\sin\varphi_r}{\sin\lambda_C}$$

and full flying range

$$\sin\beta_C = \frac{\cos\varphi_{rC}\sin\lambda_C}{\sin\psi_{0C}}, \quad L = R\beta_C.$$

Lateral deviation from the sighting plane can be defined by formulas

$$\sin\xi_C = \sin\beta_C\sin(\psi_{0C} - \psi),$$

$$z_C = R\xi_C.$$

In the case of the calculation of a trajectory for the purpose of determination of mean flying characteristics, it is possible to use the system of equations (15.12):

$$\left.\begin{aligned} \frac{dv_x}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_x - \left(\frac{g}{r} - \frac{2}{3}\omega_3^2\right)x,\\ \frac{dv_y}{dt} &= -kc_x\frac{\rho}{\rho_0}vv_y - \left(\frac{g}{r} - \frac{2}{3}\omega_3^2\right)(R+y),\\ \frac{dx}{dt} &= v_x,\\ \frac{dy}{dt} &= v_y \end{aligned}\right| \qquad (28.2)$$

Speed and angle θ are determined by the formula

$$\operatorname{tg}\theta = \frac{v_y}{v_x},$$

$$v = \frac{v_x}{\cos\theta}.$$

Altitude above the surface of Earth can be found by the formula

$$h = y + \Delta h,$$

where $\Delta h = \frac{x^2}{2R}$ is determined depending upon coordinate x. It is possible also to calculate the altitude as

$$h = \sqrt{(R+y)^2 + x^2} - R.$$

During calculations of trajectories for a very great distance it is more convenient to use the system of equations of motion in polar coordinates (15.10):

$$\left.\begin{aligned} \ddot r - r\dot\chi^2 &= -kc_x\frac{\rho}{\rho_0}v\dot r - g + \frac{2}{3}\omega_3^2 r,\\ r\ddot\chi + 2\dot r\dot\chi &= -kc_x\frac{\rho}{\rho_0}vr\dot\chi. \end{aligned}\right| \qquad (28.3)$$

Introducing designations $s = r^2\dot{\chi}$ – doubled areal velocity of the rocket with respect to the center of earth and $l = R\chi$ – distance along the arc of earth's surface, it is easy to obtain from the system of equations (28.3) the system

$$\left.\begin{aligned} &\frac{ds}{dt} = -kc_x\frac{\rho}{\rho_0}vs,\\ &\frac{dv_r}{dt} = \frac{d^2r}{dt^2} = \frac{s^2}{r^3} - kc_x\frac{\rho}{\rho_0}vv_r - \left(g - \frac{2}{3}\omega_з^2 r\right),\\ &\frac{dr}{dt} = v_r,\\ &\frac{dl}{dt} = \frac{Rs}{r^2}. \end{aligned}\right| \tag{28.4}$$

The speed and angle of inclination of the tangent are determined from the relationships

$$\left.\begin{aligned} &\mathrm{tg}\,\theta = \frac{v_r r}{s},\\ &v = \frac{s}{r\cos\theta}. \end{aligned}\right| \tag{28.5}$$

The values $\frac{\rho}{\rho_0}$, $\sqrt{\frac{T_0}{T}}$ and g necessary for calculation are taken from tables depending upon the altitude. It is usefully also to have the table of values of the quantity $g - \frac{2}{3}\omega_з^2 r$, also dependent on altitude.

§ 29. Use of Electronic Computers for Check Calculations

Not touching upon the technology of the programming of problems for carrying out ballistic calculations, we very briefly will dwell on certain distinctions between manual and machine calculation.

This, first of all, is the method of assignment of different variables, which with hand calculation are assigned either graphs (for example, aerodynamic and centering characteristics) or tables (for example, parameters of the atmosphere) allowing linear interpolation. In principle it is possible in machine calculation to use tables which permit managing only by linear interpolation. However, such tables are bulky, occupy impermissibly large capacity in the operative storage of the machine, or, being placed in devices of external memory, require frequent appeals to these devices, and thereby sharply reduce the rate of work leading to unproductive expenditures of machine time. Furthermore, the preparation and input of these data into the machine also requires additional rather long time.

The most widespread, therefore, is the method of assignment of similar dependenses with help of polynomials. An approximated curve is divided into series of sections, each of which can be represented in the form of a polynomial (usually third degree) with required accuracy. Neighboring sections should give with equal arguments equal values of the function and its first derivative.

The most convenient form of the recording of such form of polynomials is:

$$y = y_1 + (y_2 - y_1)(3 - 2\xi)\xi^2 + \xi(1 - \xi)[(1 - \xi)\eta_1 - \xi\eta_2], \tag{29.1}$$

where

$$\xi = \frac{x - x_1}{x_2 - x_1},$$

$$\eta_1 = \left(\frac{dy}{d\xi}\right)_1 = \left(\frac{dy}{dx}\right)_1 (x_2 - x_1),$$

$$\eta_2 = \left(\frac{dy}{d\xi}\right)_2 = \left(\frac{dy}{dx}\right)_2 (x_2 - x_1);$$

y_1 and y_2 are values of the approximated function on ends of the sections (i.e., when $x = x_1$ and $x = x_2$), and $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$ are derivatives at the same points.

If for some reasons it is impossible to calculate directly the values of derivatives on the ends, then it is possible to use the following method of their determination. We divide the section into three equal parts and take from the graph values y_1, y_2, y_3, and y_4 corresponding to x_1, x_2, x_3, and x_4. Values of derivatives $\left(\frac{dy}{d\xi}\right)_1$ and $\left(\frac{dy}{d\xi}\right)_4$, with the condition of passage of the polynomial through all four points, are determined by the formulas

$$\eta_1 = \left(\frac{dy}{d\xi}\right)_1 = \frac{-11y_1 + 18y_2 - 9y_3 + 2y_4}{2},$$

$$\eta_4 = \left(\frac{dy}{d\xi}\right)_4 = \frac{-2y_1 + 9y_2 - 18y_3 + 11y_4}{2}.$$

Here

$$\xi = \frac{x - x_1}{x_4 - x_1}.$$

It is possible to divide the section into four parts, as was shown on Fig. 29.1.

From the condition of passage of the polynomial through the designated four points, it is possible to write for derivatives on ends of expression

$$\eta_1 = \left(\frac{dy}{d\xi}\right)_1 = \frac{-19y_1 + 24y_2 - 8y_3 + 3y_4}{3},$$

$$\eta_4 = \left(\frac{dy}{d\xi}\right)_4 = \frac{-3y_1 + 8y_2 - 24y_3 + 19y_4}{3}.$$

The process of the selection of coefficients and partition on the necessary quantity of sections can also be assigned to the machine. This operation is fulfilled beforehand, and with the basic calculation of the trajectory there are used sections and coefficients already selected by the described method.

In certain cases it is more convenient to use not the approximating polynomials but additional differential equations integrable in parallel with the basic system of equations. For example, instead of determining the pressure and density of the atmosphere from tables or with the help of expressions (4.3) and (4.4) containing the integrals, it is possible to use differential equation (4.2), which is reduced to the form

$$\frac{dp}{dt} = -g\rho\frac{dh}{dt} = -\frac{gp}{RT}\frac{dr}{dt}.$$

The derivative dr/dt, using relationship (1.2), can be replaced by the expression

$$\frac{dr}{dt} = \frac{1}{r}\left[(R + y)\frac{dy}{dt} + x\frac{dx}{dt} + z\frac{dz}{dt}\right] = \frac{xv_x + (R + y)v_y + zv_z}{r}.$$

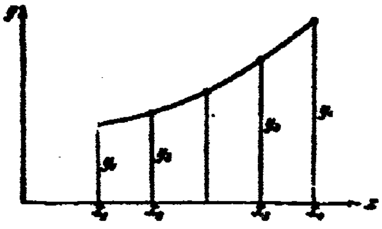

Fig. 29.1.

so that

$$\frac{dp}{dt}=-\frac{gp}{RT}\,\frac{xv_x+(R+y)v_y+zv_z}{r}. \qquad (29.2)$$

This equation should be joined to the system of differential equations of motion. An increase in the order of the system does not cause fundamental difficulties during its numerical integration, but it is simpler to calculate the right side of equation (29.2) than to find value p with the help of approximating polynomials or from expression (4.3). Function T = T(h) entering into equation (29.2) consists of several linear sections, and therefore is also very simply calculated. Air density ρ is found with help of expression

$$\frac{\rho}{\rho_0}=\frac{T_0}{T}\,\frac{p}{p_0}.$$

However, density in equations of motion is present only in the expression for impact pressure

$$q=\frac{\rho v^2}{2}, \qquad (29.3)$$

which can be transformed in the following way:

$$q=\frac{1}{2}\rho(aM)^2,$$

where M is the Mach number and a, the speed of sound in air, which is expressed by the formula

$$a=\sqrt{k\frac{p}{\rho}},$$

k = 1.405 is the ratio of heat capacities. Thus,

$$q=\frac{k}{2}\,pM^2. \qquad (29.4)$$

Since p and M are used in other members of equations of motion, expression (29.4) is more preferable for impact pressure than (29.3).

For integration of equations of motion it is convenient to use the method of Runge-Kutta. Selection of the pitch of integration depends on problems formulated before the calculation. If only final results are important then it is better to use the automatic selection of pitch, assigning the required accuracy of calculation. If, however, it is required to obtain all elements of the trajectory with respect to time of flight, then it is better to conduct integration with the assigned constant pitch, which should not be greater than that permissible from conditions of the assigned accuracy of the calculation.

It is recommended to integrate in parallel the apparent acceleration of the rocket (load factor multiplied by g_0) in the projection on the longitudinal axis, since the corresponding integral (apparent speed) in many cases is the tuning value for the automatic range control machine.[1]

Machine calculation is very convenient in the solution of boundary value problems. Most frequently encountered is the problem by definition of the azimuth of sighting and moment of turning off of the engine (i.e., initial conditions), providing a hit at the point of the earth's surface with the assigned coordinates, rectangular or, more frequently, geographic. Of the number of possible methods of the solution of the boundary value problems we will explain only one, founded, essentially, on the successive approximation to the assigned boundary conditions with the help of the assumption of the linear dependence between assigned coordinates and initial conditions.

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[1] Concerning the flight range control see § 36 and § 37 of part three.

Let us designate φ_0 and λ_0 – the assigned latitude and longitude of the point of aiming; t_0 and ψ_0 – unknown initial conditions, i.e., time of turning off of the engine and azimuth of the direction of aiming. In the beginning calculation is conducted of a certain reference trajectory at some values $t = \bar{t}$ and $\psi = \bar{\psi}$ and corresponding values $\bar{\varphi}$ and $\bar{\lambda}$ are determined; then there are computed two trajectories, each of which differs from the reference because of some deviation, Δt or $\Delta\psi$. Ratios of obtained deviations $\Delta\varphi$ and $\Delta\lambda$ to deviations Δt or $\Delta\psi$ are taken for the corresponding derivatives

$$\frac{\partial\varphi}{\partial t}, \quad \frac{\partial\lambda}{\partial t}, \quad \frac{\partial\varphi}{\partial\psi}, \quad \frac{\partial\lambda}{\partial\psi}.$$

Further, assuming the dependence between assigned coordinates and initial data to be linear in the whole interval, we determine what should be the corrections of Δt_1 and $\Delta\psi_1$ for $\bar{t}$ and $\bar{\psi}$ in order to fall into the assigned point. For this we solve the system from two equations:

$$\Delta\varphi_1 = \varphi_0 - \bar{\varphi} = \frac{\partial\varphi}{\partial t}\Delta t_1 + \frac{\partial\varphi}{\partial\psi}\Delta\psi_1,$$
$$\Delta\lambda_1 = \lambda_0 - \bar{\lambda} = \frac{\partial\lambda}{\partial t}\Delta t_1 + \frac{\partial\lambda}{\partial\psi}\Delta\psi_1.$$

Let us take new values $t_1 = t + \Delta t_1$ and $\psi_1 = \bar{\psi} + \Delta\psi_1$ and again calculate the trajectory and obtain coordinates of the point of impact φ_1 and λ_1. Again taking this obtained trajectory as a reference, the whole cycle of calculations is repeated. The process continues until we obtain coordinates of the point of impact with the assigned accuracy.

§ 30. Compilation of Preliminary Tables of Firing

Preliminary tables of firing are compiled by calculation data and contain basic values by which setting of instruments controlling distance is produced. Preliminary tables of firing are used with the conducting of experimental firing from the assigned point of launch according to the assigned direction. Therefore, before we proceed to their composition, it is necessary to know the latitude of the point of launch and azimuth of firing. Regarding the method of compilation of these tables, it consists in the following.

From the most exact equations of motion which can be used for calculation of the trajectory in accordance with the presence of initial data calculation is produced of the powered-flight trajectory.

For the calculation of sections of free flight there is selected a series of moments of the turning off of the engine. The points of turning off are characterized by elements of the trajectory t_{Ki}, x_{Ki}, y_{Ki}, v_{Ki}, θ_{Ki}, where $i = 1, 2, \ldots,$ n is from the quantity of selected reference points.

Calculations are made for n free sections and for each of them the following are determined: L – full range; C_i – setting of the instrument controlling the range (turning off the engine), and other interesting characteristics of the trajectory, for example, h_{Bi} – maximum altitude of the trajectory; v_{Bi} – speed at peak of the trajectory; v_{ci} – speed at point of collision; T_i – full time of flight, and so forth. Taken as the basic parameter depending upon which other values contained in tables of firing will be determined is the range L or setting of instrument C.

With the help of one of common methods of interpolation basic elements of the trajectory for any intermediate values of L or C are determined. For example, if there is used the Lagrange interpolation formula

$$y = y_1 \frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} + y_2 \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} + \dots + y_n \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})},$$

then into it are substituted:

instead of y_1, y_2, ..., y_n – values of some element obtained as a result of the calculation of n trajectories;

instead of x_1, x_2, ..., x_n – values of L (or C) obtained as a result of the calculation of n trajectories;

instead of x – the intermediate values of L (or C) for which it is desirable to determine other elements of the trajectory contained in the preliminary tables of firing.

It is not recommended to take number of reference points n too large (above n = 3-4). Even if there is calculated a great number of reference trajectories, then for interpolation one should use not all the obtained data but only data from three-four reference trajectories nearest to that for which interpolation is produced.

In the described method of compilation of preliminary tables of firing which are usually used during manual calculation, the selective reference points are, in general, arbitrary, and it is necessary only to see to it that they more or less evenly cover the whole assigned range of distances.

In the use of electronic computers there is the possibility of solving several boundary problems from a number of assumed purposes and determining all the necessary adjusting data for instruments and also flight paths precisely for these purposes.

It is useful to supply preliminary tables of firing by tables of corrections, which allow considering the influence of small changes of design characteristics of the rocket and sighting data on the flight path of the rocket, in particular, on coordinates of the point of impact. Methods of calculation of such corrections are examined in the next chapter.

PART THREE

DISPERSION WITH FIRING BY LONG-RANGE ROCKETS

CHAPTER VIII

FORMULATION OF THE PROBLEM

§ 31. Certain Information From the Probability Theory

As is known, if quantity u is a linear function of independent random quantities x, y, ..., t

$$u = ax + by + \dots + kt, \tag{31.1}$$

where these values have normal distribution with mean values $\bar{x}$, $\bar{y}$, ..., $\bar{t}$ and standard deviations σx, σy, ..., σt, then quantity u also has normal distribution with the mean value of $\bar{u}$ equal to

$$\bar{u} = a\bar{x} + b\bar{y} + \dots + k\bar{t}$$

and with standard deviation σu equal to

$$\sigma u = \sqrt{(a\sigma x)^2 + (b\sigma y)^2 + \dots + (k\sigma t)^2}. \tag{31.2}$$

In the theory of firing dispersion frequently is characterized by the probable (mean) deviation B, connected with the standard deviation by relationship B = 0.6745σ. For the maximum deviation Δ there is usually taken such a value that the probability p of obtaining greater (in absolute value) deviations is quite small. This value is also connected by certain constants proportionality factor with the standard deviation σ. Thus when Δ = 4B = 2.698σ ≈ 2.7σ the probability p is equal to 0.007, and when Δ = 3σ p = 0.003. Thus the maximum deviation is a conditional concept, but rather convenient for practical purposes if one were to thoroughly remember its meaning. For probable and maximum deviation of random variable u formulas being the result of formula (31.2) are correct:

$$Bu = \sqrt{(aBx)^2 + (bBy)^2 + \dots + (kBt)^2}$$

and

$$\Delta u = \sqrt{(a\Delta x)^2 + (b\Delta y)^2 + \dots + (k\Delta t)^2}.$$

It follows from this that if u is the common function u = f(x, y, ..., t) of independent values x, y, ..., t, obeying the normal law of distribution, and the maximum deviations Δx, Δy, ..., Δt are so small that partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots, \frac{\partial f}{\partial t}$$

can be considered constant when

$$\begin{gathered}\bar{x}-\Delta x<x<\bar{x}+\Delta x,\\ \bar{y}-\Delta y<y<\bar{y}+\Delta y,\\ \cdots\cdots\cdots\cdots\\ \bar{t}-\Delta t<t<\bar{t}+\Delta t,\end{gathered}$$

then the mean value u can be determined by the formula

$$\bar{u}=u_0+\frac{\partial f}{\partial x}(\bar{x}-x_0)+\frac{\partial f}{\partial y}(\bar{y}-y_0)+\ldots+\frac{\partial f}{\partial t}(\bar{t}-t_0), \qquad (31.3)$$

where x_0, y_0, ..., t_0 are certain fixed (nominal) values of quantities x, y, ..., t, quite close to the mean values these quantities $\bar{x}$, $\bar{y}$, ..., $\bar{t}$ (so that $|x_0-\bar{x}|<\Delta x$ and so forth), and $u_0=f(x_0, y_0, \ldots, t_0)$.

The root-mean-square, mean and maximum deviation of quantity u are expressed by the following formulas:

$$\sigma u=\sqrt{\left(\frac{\partial f}{\partial x}\sigma x\right)^2+\left(\frac{\partial f}{\partial y}\sigma y\right)^2+\ldots+\left(\frac{\partial f}{\partial t}\sigma t\right)^2}, \qquad (31.4)$$

$$Bu=\sqrt{\left(\frac{\partial f}{\partial x}Bx\right)^2+\left(\frac{\partial f}{\partial y}By\right)^2+\ldots+\left(\frac{\partial f}{\partial t}Bt\right)^2}, \qquad (31.5)$$

$$\Delta u=\sqrt{\left(\frac{\partial f}{\partial x}\Delta x\right)^2+\left(\frac{\partial f}{\partial y}\Delta y\right)^2+\ldots+\left(\frac{\partial f}{\partial t}\Delta t\right)^2}. \qquad (31.6)$$

In formulas (31.3)-(31.6) there is a difference between the systematic deviations and standard, mean and maximum deviations. If systematic deviations are added, as formula (31.3) shows, according to the law

$$\bar{u}-u_0=\frac{\partial f}{\partial x}(\bar{x}-x_0)+\frac{\partial f}{\partial y}(\bar{y}-y_0)+\ldots+\frac{\partial f}{\partial t}(\bar{t}-t_0)$$

(in similar cases will say that quantities are added algebraically), then the standard, mean and maximum deviations are added according to the law expressed by formulas (31.4)-(31.6). We will say that such values are added geometrically. Let us note that separate probable deviations are added algebraically:

$$u-\bar{u}=\frac{\partial f}{\partial x}(x-\bar{x})+\frac{\partial f}{\partial y}(y-\bar{y})+\ldots+\frac{\partial f}{\partial t}(t-\bar{t}).$$

§ 32. Formulation of the Problem of Dispersion

During firing by long-range rockets there appear both accidental and systematic deviations. Consequently, actual trajectories of the rockets differ from the calculation and for every rocket released differ in their own way.

What are the causes of the deviation of the trajectory of the rocket from the calculation?

First, a whole series of constants entering into the equation motion actually has values distinguished from those which are accepted during calculation. The most important of these values are the following: initial weight of the rocket, nominal thrust of the engine on earth determined by the specific thrust and flow rate per second, adjusting data of control instruments, parameters of the atmosphere on earth, and so forth.

Secondly, the actual laws of the change in the number of quantities from law accepted during calculation. Such laws are the accretions of thrust and flow rate per second with the switching on of the engine and decrease in these quantities with the turning off, changes of flow rate per second in flight, changes of aerodynamic coefficients, changes of the angle of inclination of the axis of the rocket, changes of parameters of the atmosphere on altitude and so forth.

Thirdly, a number of factors, in general, is not considered in equations of motion. Examples can serve as perturbing forces and moments appearing as a result of geometric asymmetry of the rocket, the presence of angles of attack during free flight of the rocket, etc.

Such a division is to certain degree conditional. For example, by methods of the theory of random processes (random functions) practically all possible forms of the law of change of any quantity can be represented with a sufficient degree of accuracy in the form of a family, depending on several independent accidental parameters. These parameters on a level with constants of the first group determine the flight of the rocket, and the influence of both those and others can be investigated by identical methods.

Regarding the third group of factors, it depends on the form of equations of motion utilized for calculation of the trajectory. In principle it is possible to write the equation of motion considering any factors whose physical manifestation is quite well-known, but these equations, in view of their cumbersomeness, by far cannot always be used for numerical calculation of the trajectory even on electronic computers. Therefore, for an appraisal of the influence of such factors on the flight and dispersion of rockets there have to be developed special methods. Usually problem is reduced to proof of the possibility to disregard these factors.

All quantities causing dispersion, be it deviations of constants from their nominal values or deviations of variables from nominal laws of their change, or causes which are not considered in equations of motion, will be called perturbing factors. Perturbing factors can be both systematic and accidental. Certain factors, for example, deviations of aerodynamic coefficients from their computed values, have chiefly a systematic character, while others, for example, deviations in specific thrust of the engine, are chiefly an accidental nature.

In deriving general equations of motion for the solution of problems of ballistics, we disregarded oscillations of the rocket with respect to the center of gravity, since they affect little the motion of the center of gravity. In the same place it was noted that the law of change of the angle of inclination of the tangent affects little the flying range.

Therefore, during the investigation of dispersion we will consider the angle of inclination of the tangent the assigned function and will assume as a basis only the first, third and fourth equations of the system (14.25), where during the calculations we will disregard the member with g_x. For the section of free flight we will use formulas of the elliptic theory. The influence of the rotation of earth will be disregarded since it leads only to systematic deviations from the trajectory, calculated neglecting this rotation, and only for very large distances is it necessary to be considered with the dependence of this deviation on the form of the perturbed trajectory.

Thus, the investigation of motion of the rocket consists of the following basic stages:

1. Preparation of initial data: determination of basic design data of the rocket, engine and control system, sxposure of perturbing factors and an appraisal of their random characteristics (mean values and standard deviations).

2. Ballistic calculation, having as its purpose to determine with a certain degree of accuracy the average motion of the rocket with about nominal values of all the design parameters, neglecting perturbing factors and oscillations of the rocket.

3. Calculation of stability of yawing motion as a result of which there is determined the influence on the flight of the rocket of those perturbing factors which cannot be introduced into the equations of motion for ballistic calculation and also of oscillations of the rocket about the center of gravity.

With calculation of the stability of yawing motion examined jointly are equations for angles of the direction of the tangent (θ and σ), equations of the motion about the center of gravity (for φ, ξ, η), and equations of control (for angles of deviation of effectors). Values of speed and coordinates of the rocket are taken from the ballistic calculation, since their deviation affect little the investigated values. When necessary deviations of speed and coordinates can be examined as perturbing factors.

4. Calculation of stability in longitudinal motion or calculation of range dispersion, the assignment of which is to determine the influence on flight of the rocket of such perturbing factors which can be clearly introduced into equations of motion, not changing the form of the latter. With this there are examined jointly equations of motion of the center of gravity of the rocket (for speed and coordinates), and the direction of tangent (angles $\hat{\theta}$ and σ) and other angulor values in which the necessity can be met are taken from the ballistic calculation, since the influence of deviations of these values on speed and the coordinates is small. If this is necessary, deviations in the direction of the tangent are introduced as perturbing factors. Of course the scheme of the calculation of range dispersion can and, in certain cases, should be complicated. But since the formulas only become somewhat bulkier, and the methods of calculation of dispersion in principle are not changed, then we will limit ourselves to this simplest scheme.

The main problem will subsequently be the analysis of questions connected with the calculation of range dispersion. We will start from the determination of the influence of small perturbing factors on the trajectory of the rocket and only at the end will establish the connection between average characteristics of dispersion of these factors and appropriate characteristics of the dispersion of rockets. Since systematic and probable deviations are added algebraically, we will not make a distinction between them until the question is about average characteristics of dispersion. This means that the obtained results can be applied not only to the investigation of dispersion but also to determination of the influence of small changes of design parameters of the rocket on its flying characteristics.

C H A P T E R IX

INFLUENCE OF SMALL PERTURBING FACTORS ON THE TRAJECTORY OF A ROCKET. CALCULATION OF DISPERSION

§ 33. Deviations on the Powered Section of the Trajectory

As a concrete example we will give the general method for investigation of the influence of small perturbing factors clearly entering into equations of motion on the trajectory of the rocket. Equations of motion for the powered section will be taken in the form

$$\left.\begin{aligned} \frac{dv}{dt} &= \frac{P-X_{1p}-X}{m} - g\sin\theta, \\ \frac{dy}{dt} &= v\sin\theta, \\ \frac{dx}{dt} &= v\cos\theta, \end{aligned}\right\} \quad (33.1)$$

where according to (10.16), (6.15), (5.23) and (3.3) when $\dot{m}$ = const

$$P = \dot{m}u' - S_a p. \quad (33.2)$$

$$X_{1p} = 4c_Q \frac{\rho_p u_p^2}{2} S_p. \quad (33.3)$$

$$X = c_x \frac{\rho v^2}{2} S. \quad (33.4)$$

$$m = m_0 - \dot{m}t. \quad (33.5)$$

Quantities $\dot{m}$, u', c_Q, u_p, S_p will be considered random, i.e., variable from rocket to rocket, but constant during the period of the powered section. We will assume, as earlier, that $\dot{m}$ and u' do not depend on each other, that the density ρ_p of the gas flow incident on the control surface is proportional to the flow rate per second $\dot{m}$, that the coefficient of drag of control surface c_Q is inversely proportional to the speed u_p of flow incident on the control surface (law c_Q = const/M is fully acceptable for small changes of the M number of the gas flow), and finally that quantity u_p is directly proportional to the fictitous exit velocity u'. Then expression (33.3) for the drag of jet vanes can be rewritten in the form

$$X_{1p} = k\dot{m}u',$$

where k is a certain constant, and the thrust after subtracting losses on the control surfaces can be represented by the expression

$$P-X_{1р}=(1-k)\dot{m}u'-S_a p=\dot{G}P_{\text{уд.п.р}}-S_a p_0\frac{p}{p_0}.$$

where

$$P_{\text{уд.п.р}}=\frac{1-k}{g_0}u'$$

is the specific thrust of the engine in a vacuum taking into account losses on control surfaces, which distinguished only by a constants factor from the fictitious exit velocity u'; $\dot{G}=g_0\dot{m}$ is the weight of the flow rate per second of fuel.

In order to consider possible deviations of the coefficient of drag c_x on the computed value, let us introduce into formula (33.4), as is accepted in ballistics, the form factor i:

$$X=ic_x\frac{\rho v^2}{2}S=\frac{i\rho_0 S}{2}c_x\frac{\rho}{\rho_0}v^2.$$

Finally, the expression for mass m will be written in such a form:

$$m=\frac{1}{g_0}(G_0-\dot{G}t).$$

Equations of motion take the form

$$\left.\begin{aligned}&\frac{dv}{dt}=g_0\frac{\dot{G}P_{\text{уд.п.р}}-S_a p_0\frac{p}{p_0}-\frac{i\rho_0 S}{2}c_x\frac{\rho}{\rho_0}v^2}{G_0-\dot{G}t}-g\sin\theta,\\&\frac{dy}{dt}=v\sin\theta,\\&\frac{dx}{dt}=v\cos\theta.\end{aligned}\right\}\qquad(33.6)$$

Let us investigate, in particular, the influence on the trajectory of small deviations of the following parameters with which will assume the designations λ_k: $\lambda_1=G_0$ — initial weight; $\lambda_2=\dot{G}$ — flow rate per second; $\lambda_3=P_{\text{уд.г.р}}$ — specific thrust; $\lambda_4=\frac{i\rho_0 S}{2}$ — coefficient in the expression for drag; $\lambda_5=S_a p_0$ — coefficient of altitude performance.

Coefficient λ_4 can change both owing to a change in the form factor i and due to the air density on earth ρ_0. Coefficient λ_5 is considered the possible change in air pressure on earth and also the change in altitude performance of the engine.

Finally, in order to consider possible deviations in the form of the trajectory from the calculation, we will consider the angle of inclination of the tangent θ variable according to the law

$$\theta=\theta_{\text{расч}}+\lambda_6+\lambda_7 t.\qquad(33.7)$$

The member λ_6 constitutes a constant deviation of the angle θ from the calculation $\theta_{\text{расч}}$, and the member $\lambda_7 t$ is a uniform departure of this angle from the calculation law of change. Nominal values of parameters λ_6 and λ_7 are equal to zero.

As an exercise the reader is offered to examine the system of equations which will be obtained if the system (33.1) supplements the equation

$$\frac{d\theta}{dt}=\frac{1}{v}\left[\frac{\varphi-\theta}{m}\left(P-X_{1\varphi}+\frac{l_1-x_2}{l_1-x_7}c'_y qS\right)-g\cos\theta\right],$$

serving for the determination of θ jointly with v, x and y.

Expression (33.7) should be replaced by expression

$$\varphi=\begin{cases}\varphi_{\text{пр}} & \text{when } t\leqslant t_1,\\ \varphi_{\text{пр}}+\lambda_6+\lambda_7(t-t_1) & \text{when } t>t_1,\end{cases}$$

where λ_6 and λ_7 have a former meaning but determine the error in the assignment of the program of the pitch angle, and t_1 is the time of the beginning of the program turn. Possible deviations from the face value of the function

$$\frac{l_1-x_2}{l_1-x_7}c'_y$$

can be disregarded. For an expanded system it is possible to make calculations fully analogous to those which will be made in this and the following paragraphs.

The general form of equations of motion (33.6) with the accepted designations is the following:

$$\left.\begin{aligned}\frac{dv}{dt}&=f_1(t,\ v,\ y,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6,\ \lambda_7),\\ \frac{dy}{dt}&=f_2(t,\ v,\ \lambda_6,\ \lambda_7),\\ \frac{dx}{dt}&=f_3(t,\ v,\ \lambda_6,\ \lambda_7),\end{aligned}\right\}\qquad(33.8)$$

where

$$\left.\begin{aligned}f_1&=g_0\frac{\lambda_2\lambda_3-\lambda_5\frac{p}{p_0}-\lambda_4 c_x\frac{\rho}{\rho_0}v^2}{\lambda_1-\lambda_3 t}-g\sin(\theta_{\text{расч}}+\lambda_6+\lambda_7 t),\\ f_2&=v\sin(\theta_{\text{расч}}+\lambda_6+\lambda_7 t),\\ f_3&=v\cos(\theta_{\text{расч}}+\lambda_6+\lambda_7 t).\end{aligned}\right\}\qquad(33.9)$$

Under initial conditions

$$v=0,\quad y=0,\quad x=0\qquad(t=0)\qquad(33.10)$$

the solution of system (33.8) has the form

$$\left.\begin{aligned}v&=\varphi_1(t,\ \lambda_1,\ \lambda_2,\ \ldots,\ \lambda_7),\\ y&=\varphi_2(t,\ \lambda_1,\ \lambda_2,\ \ldots,\ \lambda_7),\\ x&=\varphi_3(t,\ \lambda_1,\ \lambda_2,\ \ldots,\ \lambda_7).\end{aligned}\right\}\qquad(33.11)$$

These expressions show that speed and coordinates of the center of gravity of the rocket depend not only on the time of flight t but also on values of parameters λ_1, λ_2, ..., λ_7. Let us investigate this dependence. At small changes of time t and parameters λ_k one can assume that

$$\left.\begin{aligned}\Delta v&=\frac{\partial\varphi_1}{\partial t}\Delta t+\sum_{k=1}^{7}\frac{\partial\varphi_1}{\partial\lambda_k}\Delta\lambda_k,\\ \Delta y&=\frac{\partial\varphi_2}{\partial t}\Delta t+\sum_{k=1}^{7}\frac{\partial\varphi_2}{\partial\lambda_k}\Delta\lambda_k,\\ \Delta x&=\frac{\partial\varphi_3}{\partial t}\Delta t+\sum_{k=1}^{7}\frac{\partial\varphi_3}{\partial\lambda_k}\Delta\lambda_k,\end{aligned}\right\}\qquad(33.12)$$

i.e., the dependence v, y and x on $\Delta\lambda_k$ will be determined as only partial derivatives $\frac{\partial\varphi_i}{\partial\lambda_k}$ will be known.

Let us introduce the designations:

$$\left.\begin{aligned}&\frac{\partial\varphi_1}{\partial t}=\dot v,\quad \frac{\partial\varphi_2}{\partial t}=\dot y,\quad \frac{\partial\varphi_3}{\partial t}=\dot x\quad \frac{\partial\varphi_i}{\partial\lambda_k}=z_{ik},\\ &\frac{\partial f_i}{\partial v}=a_{i1},\quad \frac{\partial f_i}{\partial y}=a_{i2},\quad \frac{\partial f_i}{\partial\lambda_k}=\beta_{ik}.\end{aligned}\right\}\qquad(33.13)$$

All these partial derivatives should be computed at nominal values of parameters of λ_k and at values v, x, y, θ taken for every value t from the calculated trajectory. In these designations the dependences (33.12) take following form:

$$\left.\begin{aligned}\Delta v&=\dot v\,\Delta t+\sum_k z_{1k}\Delta\lambda_k,\\ \Delta y&=\dot y\,\Delta t+\sum_k z_{2k}\Delta\lambda_k,\\ \Delta x&=\dot x\,\Delta t+\sum_k z_{3k}\Delta\lambda_k.\end{aligned}\right\}\qquad(33.14)$$

Quantities $\dot v$, $\dot y$, $\dot x$ are determined with integration of the system (33.8). In order to obtain a system of equations to determine values z_{ik} interesting to us, we will differentiate equations of the system (33.8) with respect to λ_k. Since under the sign of functions f_i on parameter λ_k depend only v, y and λ_k, we will obtain:

$$\begin{aligned}\frac{\partial}{\partial\lambda_k}\left(\frac{dv}{dt}\right)&=\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial\lambda_k}+\frac{\partial f_1}{\partial y}\cdot\frac{\partial y}{\partial\lambda_k}+\frac{\partial f_1}{\partial\lambda_k},\\ \frac{\partial}{\partial\lambda_k}\left(\frac{dy}{dt}\right)&=\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial\lambda_k}+\frac{\partial f_2}{\partial\lambda_k},\\ \frac{\partial}{\partial\lambda_k}\left(\frac{dx}{dt}\right)&=\frac{\partial f_3}{\partial v}\cdot\frac{\partial v}{\partial\lambda_k}+\frac{\partial f_3}{\partial\lambda_k}.\end{aligned}$$

Let us change the order of differentiation in the left sides of these equations and use designations of (33.13):

$$\left.\begin{aligned}\frac{dz_{1k}}{dt}&=a_{11}z_{1k}+a_{12}z_{2k}+\beta_{1k},\\ \frac{dz_{2k}}{dt}&=a_{21}z_{1k}+\beta_{2k},\\ \frac{dz_{3k}}{dt}&=a_{31}z_{1k}+\beta_{3k}.\end{aligned}\right\}\qquad(33.15)$$

This system of linear differential equations with definite assumptions (existence and continuity of partial derivatives α_{ij} and β_{ik}, i, j = 1, 2, 3, k = 1, 2, ..., 7) can serve to determine quantities z_{ik}. Strict proof of this affirmation can be found in the general theory of systems of differential equations.

Since no changes of parameters λ_k can influence the initial values (33.10) of functions v, y, x the initial conditions for integration of the system (33.15) will be $z_{ik} = 0$ when $t = 0$.

In order to find evident expressions for partial derivatives α_{ij} and β_{ik}, we will differentiate expression (33.9) for functions f_i with respect to v, y and λ_k, and for the simplification of results of differentiation we will use the formulas (in which M_3 and R_3 denote the mass and radius of earth):

$$M = \frac{v}{a},$$

$$a = a_0\sqrt{\frac{T}{T_0}},$$

$$p = \rho RT,$$

$$\frac{dp}{dy} = -g\rho,$$

$$g = \frac{fM_3}{r^2},$$

$$r = R_3 + y.$$

Let us also take into account that the coefficient of drag c_x depends on the M number and on altitude y and the absolute air temperature T only on altitude.

After rather long, but not complicated, computations there can be obtained the following expressions:

$$\alpha_{11} = \frac{\partial f_1}{\partial v} = -\frac{S\rho v}{m}\left(c_x + \frac{M}{2}\frac{\partial c_x}{\partial M}\right). \tag{33.16}$$

$$\alpha_{12} = \frac{\partial f_1}{\partial y} = \frac{\rho}{m}\left[S_a g + \frac{1}{2}Sv^2\left(\frac{gc_x}{RT} + \right.\right.$$
$$\left.\left. + \frac{1}{T}\frac{dT}{dy}\left(c_x + \frac{M}{2}\frac{\partial c_x}{\partial M}\right) - \frac{\partial c_x}{\partial y}\right)\right] + \frac{2g}{r}\sin\theta. \tag{33.17}$$

$$\alpha_{21} = \frac{\partial f_2}{\partial v} = \sin\theta, \tag{33.18}$$

$$\alpha_{31} = \frac{\partial f_3}{\partial v} = \cos\theta, \tag{33.19}$$

$$\beta_{11} = \frac{\partial f_1}{\partial \lambda_1} = -\frac{P - X_{tp} - X}{g_0 m^2}. \tag{33.20}$$

$$\beta_{12} = \frac{\partial f_1}{\partial \lambda_2} = \frac{P_{yz.\,n.\,p}}{m} + \frac{t}{g_0 m^2}(P - X_{tp} - X). \tag{33.21}$$

$$\beta_{13} = \frac{\partial f_1}{\partial \lambda_3} = \frac{\dot{G}}{m}. \tag{33.22}$$

$$\beta_{14} = \frac{\partial f_1}{\partial \lambda_4} = -\frac{2X}{\rho_0 S m}. \tag{33.23}$$

$$\beta_{15} = \frac{\partial f_1}{\partial \lambda_5} = -\frac{1}{m}\frac{p}{p_0}. \tag{33.24}$$

$$\beta_{16} = \frac{\partial f_1}{\partial \lambda_6} = -g\cos\theta. \tag{33.25}$$

$$\beta_{17} = \frac{\partial f_1}{\partial \lambda_7} = -gt\cos\theta. \tag{33.26}$$

$$\beta_{21} = \beta_{22} = \beta_{23} = \beta_{24} = \beta_{25} = 0. \tag{33.27}$$

$$\beta_{26} = v\cos\theta, \tag{33.28}$$

$$\beta_{27} = vt\cos\theta, \tag{33.29}$$

$$\beta_{31} = \beta_{32} = \beta_{33} = \beta_{34} = \beta_{35} = 0, \tag{33.30}$$

$$\beta_{36} = -v\sin\theta, \tag{33.31}$$

$$\beta_{37} = -vt\sin\theta. \tag{33.32}$$

Values of quantities

$$c_x + \frac{M}{2}\frac{\partial c_x}{\partial M}, \quad \frac{\partial c_x}{\partial y}, \quad \frac{g}{RT}, \quad \frac{1}{T}\frac{\partial T}{\partial y}$$

are calculated with the help of tables or graphs of aerodynamic coefficients of the given rocket and tables of the standard atmosphere [4]. Remaining quantities necessary for calculation of coefficients (33.16) to (33.32) are taken from ballistic calculation.

After determination of all the necessary coefficients of quantity z_{ik} systems (33.15) are calculated by means of numerical integration. When these values are found, determination of the influence of small deviations of parameters λ_k on a powered-flight trajectory is reduced to the use of formulas (33.14).

§ 34. Deviations of the Point of Turning Off of the Engine

Applying equation (33.14) to the point of turning off of the engine, we will obtain the connection between deviations of the time of turning off of the engine, speed and coordinates at the time of the turning off and the deviations of parameters λ_k

$$\left.\begin{aligned} \Delta v_1 &= \dot{v}_1 \Delta t_1 + \sum_k z_{1k}\Delta\lambda_k, \\ \Delta y_1 &= \dot{y}_1 \Delta t_1 + \sum_k z_{2k}\Delta\lambda_k, \\ \Delta x_1 &= \dot{x}_1 \Delta t_1 + \sum_k z_{3k}\Delta\lambda_k. \end{aligned}\right\} \tag{34.1}$$

But these three formulas for the determination of four deviations Δt_1, Δv_1, Δy_1 and Δx_1 are insufficient. The inadequate relationship can be obtained proceeding from the equation of operation of the instrument controlling the turning off of the engine. Let us consider the following three methods of the turning off of the engine:

1) turning off at an assigned moment of time, considering from the moment of launch;

2) turning off with achievement by the rocket of a set value of speed;

3) turning off from an integrator of G-forces.

With the turning off in time the deviation of speed and coordinates at the point of turning off of the engine is determined by formulas (34.1), in which instead of Δt_1 it is necessary to insert the instrumental error $\Delta t_{\text{и}}$ of the timing mechanism sending the command for the turning off:

$$\left.\begin{aligned} \Delta t_1 &= \Delta t_{\text{и}}, \\ \Delta v_1 &= \dot{v}_1 \Delta t_{\text{и}} + \sum z_{1k}\Delta\lambda_k, \\ \Delta y_1 &= \dot{y}_1 \Delta t_{\text{и}} + \sum z_{2k}\Delta\lambda_k, \\ \Delta x_1 &= \dot{x}_1 \Delta t_{\text{и}} + \sum z_{3k}\Delta\lambda_k. \end{aligned}\right\} \tag{34.2}$$

In the case of the turning off with speed the deviation of speed at the time of turning off from the assigned value constitutes an instrumental error $\Delta v_{\text{и}}$ of the instrument measuring speed:

$$\Delta v'_1 = \Delta v_{\text{и}}.$$

The prime will denote quantities pertaining to the turning off of the engine with speed.

By formulas (34.1) we find the deviations of remaining quantities characterizing the point of turning off of the engine:

$$\Delta t'_1 = \frac{1}{\dot v_1}\Delta v_{\text{и}} - \sum \frac{z_{1k}}{\dot v_1}\Delta\lambda_k,$$

$$\Delta y'_1 = \frac{\dot y_1}{\dot v_1}\Delta v_{\text{и}} + \sum\left(z_{2k} - \frac{\dot y_1}{\dot v_1} z_{1k}\right)\Delta\lambda_k,$$

$$\Delta x'_1 = \frac{\dot x_1}{\dot v_1}\Delta v_{\text{и}} + \sum\left(z_{3k} - \frac{\dot x_1}{\dot v_1} z_{1k}\right)\Delta\lambda_k,$$

or

$$\left.\begin{aligned} \Delta t'_1 &= \frac{1}{\dot v_1}\Delta v_{\text{и}} + \sum z'_{0k}\,\Delta\lambda_k,\\ \Delta v'_1 &= \Delta v_{\text{и}},\\ \Delta y'_1 &= \frac{\dot y_1}{\dot v_1}\Delta v_{\text{и}} + \sum z'_{2k}\,\Delta\lambda_k,\\ \Delta x'_1 &= \frac{\dot x_1}{\dot v_1}\Delta v_{\text{и}} + \sum z'_{3k}\,\Delta\lambda_k,\end{aligned}\right\} \tag{34.3}$$

where there is designated

$$\left.\begin{aligned} z'_{0k} &= -\frac{z_{1k}}{\dot v_1},\\ z'_{2k} &= z_{2k} - \frac{\dot y_1}{\dot v_1} z_{1k} = z_{2k} + \dot y_1 z'_{0k},\\ z'_{3k} &= z_{3k} - \frac{\dot x_1}{\dot v_1} z_{1k} = z_{3k} + \dot x_1 z'_{0k}.\end{aligned}\right\} \tag{34.4}$$

Before we derive a formula for deviations at the point of turning off of the engine with a turning off from the integrator, let us examine the somewhat simplified theory of this instrument.

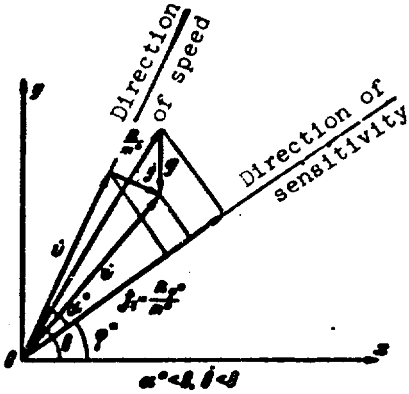

Fig. 34.1.

Let us consider the material particle connected with the body of the rocket. Disregarding rotation of the rocket about the center of gravity, we will consider that the acceleration of this material particle is equal to the acceleration of the center of gravity of the rocket. But the acceleration of the examined point is created by two forces: gravity m^*g (m^* is the mass of the point) and the force R having an effect on the point from the side of the rocket (Fig. 34.1).

Let us write the equation of motion of the point in projection on a certain direction forming the angle α^* with the tangent to the trajectory of center of the gravity of the rocket and the angle φ^*, equal to

$$\varphi^* = \theta + \alpha^*. \tag{34.5}$$

with the horizon (as the horizon we take the Ox axis, and we consider the direction of gravity parallel to the Oy axis). This equation has the form

$$m^*(\dot{v}\cos\alpha^* + v\dot{\theta}\sin\alpha^*) = R_{\varphi^*} - m^* g\sin\varphi^*. \tag{34.6}$$

where $\dot{v}\cos\alpha^*$ is the projection of the tangent acceleration of the center of gravity of the rocket on the examined direction; $v\dot{\theta}\sin\alpha^*$ — projection of the normal acceleration of the center of gravity of the rocket on this direction; R_{φ^*} — projection of force R on the same direction; $-m^* g\sin\varphi^*$ — projection of gravity on the same direction. Such a material particle is a sensing device of every integrator.

Force R_{φ^*} causes some physical effect whose action is integrated during the period of the entire powered section. For example, in one of the constructions of the integrator, the gyroscope, whose center of masses does not coincide with the center of suspension (gyroscopic pendulum), precesses under the action of force R_{φ^*} with an angular velocity, proportional to this force. The measured value is the angle of precession.

Thus the integrator produces a magnitude proportional to the integral

$$\int_0^t R_{\varphi^*}\,dt,$$

or, since mass m* remains constant, the integral

$$v_s = \int_0^t \frac{R_{\varphi^*}}{m^*}\,dt. \tag{34.7}$$

This integral will be called apparent speed and the integrand quantity

$$j_s = \frac{R_{\varphi^*}}{m^*} = \frac{R_{\varphi^*}}{G^*} g_0 \tag{34.8}$$

the apparent acceleration. Apparent acceleration is nothing else but a G-force in the direction φ* multiplied by g_0. The direction determined by angle φ*, along which occurs integration of the G-force is called the direction of sensitivity of the integrator.

Substituting R_{φ^*} from equation (34.6) into formula (34.7), we will obtain

$$v_s = \int_0^t (\dot{v}\cos\alpha^* + v\dot{\theta}\sin\alpha^* + g\sin\varphi^*)\,dt =$$

$$= \int_0^t \dot{v}\cos\alpha^*\,dt + \int_0^t v\dot{\theta}\sin\alpha^*\,dt + \int_0^t g\sin\varphi^*\,dt.$$

We integrate the first component by parts:

$$\int_0^t \dot{v}\cos\alpha^*\,dt = v\cos\alpha^*\Big|_0^t - \int_0^t v\,d(\cos\alpha^*).$$

Since v = 0 when t = 0, then

$$\int_0^t \dot{v}\cos\alpha^*\,dt = v\cos\alpha^* + \int_0^t v\sin\alpha^*\dot{\alpha}^*\,dt$$

and

$$v_s = v\cos\alpha^\circ + \int_0^t v(\dot{\theta} + \dot{\alpha}^\circ)\sin\alpha^\circ\, dt + \int_0^t g\sin\varphi^\circ\, dt.$$

But on the basis of equality (34.5)

$$\dot{\varphi}^\circ = \dot{\theta} + \dot{\alpha}^\circ,$$

therefore

$$v_s = v\cos\alpha^\circ + \int_0^t v\dot{\varphi}^\circ \sin\alpha^\circ\, dt + \int_0^t g\sin\varphi^\circ\, dt. \tag{34.9}$$

In particular, when $t = t_1$

$$v_{s1} = v_1\cos\alpha_1^\circ + \int_0^{t_1} g\sin\varphi^\circ\, dt + \int_0^{t_1} v\dot{\varphi}^\circ \sin\alpha^\circ\, dt.$$

We will limit ourselves now to the consideration of the integrator rigidly secured on board the rocket so that the direction of sensitivity coincides with the direction of the longitudinal axis of the rocket. With this

$$\varphi^\circ = \varphi, \quad \alpha^\circ = \alpha,$$

$$v_{s1} = v_1\cos\alpha_1 + \int_0^{t_1} g\sin\varphi\, dt + \int_0^{t_1} v\dot{\varphi}\sin\alpha\, dt. \tag{34.10}$$

The last member is small, since the angular velocity of inclination of the axis of the rocket $\dot{\varphi}$ and angle of attack α are small. In the first member $\cos\alpha_1$ is close to unity. For an appraisal of the dispersion it is possible to use the approximate formula

$$v_{s1} = v_1 + \int_0^{t_1} g\sin\varphi\, dt, \tag{34.11}$$

or

$$v_1 = v_{s1} - \int_0^{t_1} g\sin\varphi\, dt. \tag{34.12}$$

With the examined method of the turning off of the engine the current value of the apparent speed is continuously compared with the assigned value to which the integrator is tuned. When these two values coincide, the instrument sends a command for turning off the engine. From formula (34.12) it is clear that deviation of terminal velocity v_1 is conditioned by the error $\Delta v_{s\text{И}}$ with which it is possible to sustain the assigned value of apparent speed v_{s1}, the deviation Δt_1 of the time of operation of the engine and deviation $\delta\varphi$ of angle φ during the period of the whole powered section. Furthermore, deviation of g is possible owing to the change in altitude, but it can be disregarded, and therefore

$$v_1 + \Delta v_1'' = v_{s1} + \Delta v_{s\text{и}} - \int_0^{t_1 + \Delta t_1''} g\sin(\varphi + \delta\varphi)\, dt.$$

We note by double prime values referring to the turning off of the engine with the help of the integrator. Considering $\cos\delta\varphi = 1$ and $\sin\delta\varphi = \delta\varphi$, we will obtain:

$$v_1+\Delta v_1^{*}=v_{s1}+\Delta v_{sa}-\int\limits_0^{t_1+\Delta t_1^{*}} g(\sin\varphi+\cos\varphi\,\delta\varphi)\,dt=$$

$$=v_{s1}+\Delta v_{sa}-\int\limits_0^{t_1} g\sin\varphi\,dt-\int\limits_0^{t_1} g\,\delta\varphi\cos\varphi\,dt-$$

$$-\int\limits_{t_1}^{t_1+\Delta t_1^{*}} g\sin\varphi\,dt-\int\limits_{t_1}^{t_1+\Delta t_1^{*}} g\,\delta\varphi\cos\varphi\,dt.$$

The last member having an order of $g\Delta t_1''\delta\varphi$ can be disregarded, and in the penultimate we can consider $g \sin \varphi$ constant in the time interval from t_1 to $t_1 + \Delta t_1''$. Then

$$v_1+\Delta v_1^{*}=v_{s1}+\Delta v_{sa}-\int\limits_0^{t_1} g\sin\varphi\,dt-$$

$$-\int\limits_0^{t_1} g\,\delta\varphi\cos\varphi\,dt-g_1\sin\varphi_1\,\Delta t_1^{*}.$$

Subtracting hence term by term the equality (34.12), we will obtain:

$$\Delta v_1^{*}=\Delta v_{sa}-\int\limits_0^{t_1} g\,\delta\varphi\cos\varphi\,dt-g_1\sin\varphi_1\,\Delta t_1^{*}.$$

We will designate the second member by $\Delta v_{S\varphi}$:

$$\Delta v_{s\varphi}=-\int\limits_0^{t_1} g\,\delta\varphi\cos\varphi\,dt. \tag{34.13}$$

Its value can be found if the law of deviation of the axis of the rocket from the calculation position is known. Finally

$$\Delta v_1^{*}=\Delta v_{sa}+\Delta v_{s\varphi}-g_1\sin\varphi_1\,\Delta t_1^{*}. \tag{34.14}$$

Substituting into the first of formulas (34.1) the expression (34.14), we obtain:

$$\Delta v_{sa}+\Delta v_{s\varphi}-g_1\sin\varphi_1\,\Delta t_1^{*}=\dot v_1\,\Delta t_1^{*}+\sum z_{1g}\,\Delta\lambda_g,$$

whence

$$\Delta t_1^{*}=\frac{1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sa}+\Delta v_{s\varphi})-\sum\frac{z_{1g}}{\dot v_1+g_1\sin\varphi_1}\Delta\lambda_g. \tag{34.15}$$

Now, inserting expression (34.15) into formulas (34.1), it is easy to find the deviation of remaining values at the point of the trajectory, where from the integrator a command is fed for turning off the engine:

$$\left.\begin{aligned}
\Delta v_1^{*} &= \frac{\dot v_1}{\dot v_1+g_1\sin\varphi_1}(\Delta \quad +\Delta v_{sq})+\\
&\qquad +\sum\left(z_{1k}-\frac{\dot v_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1}\right)\Delta\lambda_k.\\
\Delta y_1^{*} &= \frac{\dot y_1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\\
&\qquad +\sum\left(z_{2k}-\frac{\dot y_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1}\right)\Delta\lambda_k.\\
\Delta x_1^{*} &= \frac{\dot x_1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\\
&\qquad +\sum\left(z_{3k}-\frac{\dot x_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1}\right)\Delta\lambda_k.
\end{aligned}\right| \qquad (34.16)$$

Let us introduce for brevity these designations

$$\left.\begin{aligned}
z_{0k}^{*} &= -\frac{z_{1k}}{\dot v_1+g_1\sin\varphi_1},\\
z_{1k}^{*} &= z_{1k}-\frac{\dot v_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1} = -g_1\sin\varphi_1 z_{0k}^{*}.\\
z_{2k}^{*} &= z_{2k}-\frac{\dot y_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1} = z_{2k}+\dot y_1 z_{0k}^{*}.\\
z_{3k}^{*} &= z_{3k}-\frac{\dot x_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1} = z_{3k}+\dot x_1 z_{0k}^{*}.
\end{aligned}\right| \qquad (34.17)$$

Then formulas (34.15) and (34.16) will take such form:

$$\left.\begin{aligned}
\Delta t_1^{*} &= \frac{1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\sum z_{0k}^{*}\Delta\lambda_k.\\
\Delta v_1^{*} &= \frac{\dot v_1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\sum z_{1k}^{*}\Delta\lambda_k.\\
\Delta y_1^{*} &= \frac{\dot y_1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\sum z_{2k}^{*}\Delta\lambda_k.\\
\Delta x_1^{*} &= \frac{\dot x_1}{\dot v_1+g_1\sin\varphi_1}(\Delta v_{sm}+\Delta v_{sq})+\sum z_{3k}^{*}\Delta\lambda_k.
\end{aligned}\right| \qquad (34.18)$$

Formulas (34.2), (34.3) and (34.18) give solution to the problem of deviations at the point of turning off the engine with different methods of turning off.

§ 35. Influence of the Process of Turning Off the Engine on Dispersion

Let us examine the section of the trajectory between the point at which moves the command for the turning off of the engine and the point where the process of turning off is finished. The time interval (t_1, t_2) between both points will be selected constant and such that with practically any possible law of decrease in thrust the process of turning off will succeed in being completed during that time.

Let us make the following assumptions:

1) turning off of the engine is graduated, i.e., after the first command fed at the initial moment t_1 of the examined interval of time there occurs only a decrease in the value of the thrust up to a certain intermediate value and only after the second command fed inside the examined interval of time does the thrust start to drop to zero;

2) the angle of attack is so small that in equations of motion it can be disregarded;

3) the change in the angle of inclination of the tangent with time is the same regardless of the law of drop in thrust;

4) towards the moment of termination of the drop in thrust the drag of the rocket is negligible.

With these assumptions equations of motion of the rocket take the form

$$\left.\begin{aligned} \frac{dv}{dt} &= \frac{P - X_{1p} - X}{m} - g\sin\theta = \frac{R_\varphi}{m} - g\sin\theta, \\ \frac{dy}{dt} &= v\sin\theta, \\ \frac{dx}{dt} &= v\cos\theta, \end{aligned}\right| \tag{35.1}$$

where R_φ is the projection of all forces having an effect on the rocket, except gravity, on the direction of the axis of the rocket. Designating the moment of the feeding of the command for complete turning off of the engine by $t_{\text{к}}$, we can write:

$$v_2 - v_1 = \int_{t_1}^{t_2} \left(\frac{R_\varphi}{m} - g\sin\theta\right) dt =$$

$$= \int_{t_1}^{t_{\text{к}}} \frac{R_\varphi}{m} dt + \int_{t_{\text{к}}}^{t_2} \frac{R_\varphi}{m} dt - \int_{t_1}^{t_2} g\sin\theta\, dt,$$

or

$$v_2 = v_1 + \int_{t_1}^{t_{\text{к}}} \frac{R_\varphi}{m} dt + \int_{t_{\text{к}}}^{t_2} \frac{R_\varphi}{m} dt - \int_{t_1}^{t_2} g\sin\theta\, dt. \tag{35.2}$$

The second and third members are the increase in speed owing to all forces except gravity on sections respectively between the two commands and after the command for complete turning off. These increases due to the great scattering in the nature of a drop of thrust are themselves subject to great scattering. For the section between the two commands (t_1, $t_{\text{к}}$) this scattering can be minimized with the proper method of feed of the second command. It is easy to verify that with our assumptions in such a way there will be the turning off from the integrator. Actually, on the basis of the first equation (35.1), the second of the above-made assumptions and formula (34.11)

$$\frac{R_\varphi}{m} = \frac{dv}{dt} + g\sin\theta = \frac{dv}{dt} + g\sin\varphi = \frac{dv_s}{dt}$$

and, consequently,

$$\int_{t_1}^{t_{\text{к}}} \frac{R_\varphi}{m} dt = v_{\varphi\text{к}} - v_{\varphi1}. \tag{35.3}$$

In virtue of formula (35.2) and the third assumption

$$\Delta v_2 = \Delta v_1 + \Delta\left(\int_{t_1}^{t_{\text{к}}} \frac{R_\varphi}{m} dt\right) + \Delta\left(\int_{t_{\text{к}}}^{t_2} \frac{R_\varphi}{m} dt\right). \tag{35.4}$$

and this means that the error in speed at time t_2 is composed of the error at time t_1 and scattering of integrals

$$\int_{t_1}^{t_{\text{к}}}\frac{R_\varphi}{m}dt \text{ and } \int_{t_{\text{к}}}^{t_2}\frac{R_\varphi}{m}dt.$$

The error in the quantity of the first integral will be minimum when the command for full turning off is fed by the instrument measuring the magnitude of this integral. Formula (35.3) shows that such an instrument is precisely the integrator of axial G-forces.

Turning to the second integral, let us note that after the command for full turning off of the engine the mass of the rocket m practically has not changed.

$$\int_{t_{\text{к}}}^{t_2}\frac{R_\varphi}{m}dt=\frac{1}{m_{\text{к}}}\int_{t_{\text{к}}}^{t_2}R_\varphi dt=\frac{I}{m_{\text{к}}}, \qquad (35.5)$$

where the letter I is designated the so-called aftereffect pulse, the total pulse of all forces (except gravity) having an effect on the rocket after feed of the command for full turning off of the engine. The main force of these forces is the thrust created by the engine due to the burning and expiration of fuel components remaining in the chamber and in fuel manifolds between the chamber and cutoff valves. Part of the aftereffect pulse is caused also by the delay in operation of the cutoff valves after feeding command for turning off the engine. Forces X_{1p} and X do not play an essential role in the process of the after effect and, according to our assumptions, turn into zero together with the thrust towards moment $t < t_2$, so that the examined integral does not depend on the selection of time t_2 provided the above-mentioned conditions are observed.

The magnitude of scattering of the second integral in flight can in no way be limited, and as for its decrease one should take care of it on land. Other things being equal, this scattering will be less the lesser the pulse of the aftereffect. The latter can be decreased owing to a decrease in thrust towards the moment of feeding a command for full switching on of the engine (this is the meaning of the graduated turning off) and also due to a faster drop in the thrust.

Integrating second and third equation (35.1), we find

$$\left.\begin{aligned} y_2&=y_1+\int_{t_1}^{t_2}v\sin\theta\,dt,\\ x_2&=x_1+\int_{t_1}^{t_2}v\cos\theta\,dt,\end{aligned}\right\} \qquad (35.6)$$

and, consequently,

$$\left.\begin{aligned} \Delta y_2&=\Delta y_1+\int_{t_1}^{t_2}\Delta v\sin\theta\,dt.\\ \Delta x_2&=\Delta x_1+\int_{t_1}^{t_2}\Delta v\cos\theta\,dt.\end{aligned}\right\} \qquad (35.7)$$

Quantities of the order of $\Delta v(t_2 - t_1)$ can be disregarded, since the duration of the process of turning off is small. But then the error in the coordinates during transition from point t_1 to point t_2 are not changed:

$$\left.\begin{aligned} \Delta y_2&=\Delta y_1.\\ \Delta x_2&=\Delta x_1.\end{aligned}\right\} \qquad (35.8)$$

These formulas, together with expression (35.4), solve the problem of deviations of the basic ballistic parameters at the end of the section of the turning off of the engine. In particular, if the command for full turning off is fed by the integrator, then on the basis of equations (35.3) and (35.5)

$$\Delta v_2 = \Delta v_1 + \Delta(v_{зк} - v_{з1}) + \Delta\left(\frac{I}{m_к}\right). \tag{35.9}$$

Formulas (35.8) and (35.9) will be recorded briefly thus:

$$\Delta v_2 = \Delta v_1 + \Delta v_{12}, \tag{35.10}$$

where

$$\Delta v_{12} = \Delta\left(\int_{t_1}^{t_к} \frac{R_\varphi}{m} dt\right) + \Delta\left(\frac{I}{m_к}\right) \tag{35.11}$$

in general, and

$$\Delta v_{12} = \Delta(v_{зк} - v_{з1}) + \Delta\left(\frac{I}{m_к}\right) \tag{35.12}$$

with turning off from the integrator.

Finally, let us note that everything discussed remains in force if for the moment t_1 we take not the moment of feeding the command for a decrease in thrust, but the moment $t_к$ of the feed of the command for full turning off of the engine. As t_2 it will be necessary to take the moment separated by the constant interval of time from $t_к$, and we will obtain

$$\Delta v_2 = \Delta v_к + \Delta v_{к2}, \tag{35.13}$$

where

$$\Delta v_{к2} = \Delta\left(\int_{t_к}^{t_2} \frac{R_\varphi}{m} dt\right) - \Delta\left(\frac{I}{m_к}\right); \tag{35.14}$$

$$\left.\begin{aligned} \Delta y_2 &= \Delta y_к, \\ \Delta x_2 &= \Delta x_к. \end{aligned}\right\} \tag{35.15}$$

Formulas (35.13)-(35.15) should be used when the second command moves independently of the first.

§ 36. Range Dispersion

In examining the range dispersion we will limit ourselves to the case when full turning off of the engine is finished during negligible drag. Then the greater part of the section of free flight will lie in practically a vacuum, and for the calculation of dispersion it is possible to use formulas of the elliptic theory. Dispersion from the influence of atmosphere at the end of the descending phase of the trajectory should be investigated specially, and we will not touch upon it.

The influence of deviations v_H, h_H and ϑ_H at the initial point of the elliptic trajectory on the range of free flight is expressed by formulas (19.42) (with replacement of r_H by $R + h_H$):

$$\left.\begin{aligned}
\frac{\partial l_\text{св}}{\partial v_\text{н}} &= \frac{4R^2}{v_\text{н}}\,\frac{(1+\operatorname{tg}^2\vartheta_\text{н})\sin^2\frac{\beta_C}{2}\operatorname{tg}\frac{\beta_C}{2}}{\nu_\text{н}\left(h_\text{н}+R\operatorname{tg}\vartheta_\text{н}\operatorname{tg}\frac{\beta_C}{2}\right)},\\
\frac{\partial l_\text{св}}{\partial h_\text{н}} &= R\,\frac{\nu_\text{н}+\frac{2R}{R+h_\text{н}}(1+\operatorname{tg}^2\vartheta_\text{н})\sin^2\frac{\beta_C}{2}\operatorname{tg}\frac{\beta_C}{2}}{\nu_\text{н}\left(h_\text{н}+R\operatorname{tg}\vartheta_\text{н}\operatorname{tg}\frac{\beta_C}{2}\right)},\\
\frac{\partial l_\text{св}}{\partial \vartheta_\text{н}} &= 2R^2\,\frac{(1+\operatorname{tg}^2\vartheta_\text{н})\left(\nu_\text{н}-2\operatorname{tg}\vartheta_\text{н}\operatorname{tg}\frac{\beta_C}{2}\right)\sin^2\frac{\beta_C}{2}}{\nu_\text{н}\left(h_\text{н}+R\operatorname{tg}\vartheta_\text{н}\operatorname{tg}\frac{\beta_C}{2}\right)},
\end{aligned}\right| \qquad (36.1)$$

where

$$\nu_\text{н}=\frac{v_\text{н}^2(R+h_\text{н})}{g_0R^2}; \qquad (36.2)$$

$\operatorname{tg}\frac{\beta_C}{2}$ is the positive root of the quadratic equation

$$\begin{aligned}
&[2R(1+\operatorname{tg}^2\vartheta_\text{н})-(2R+h_\text{н})\nu_\text{н}]\operatorname{tg}^2\frac{\beta_C}{2}-\\
&\qquad -2\nu_\text{н}R\operatorname{tg}\vartheta_\text{н}\operatorname{tg}\frac{\beta_C}{2}-h_\text{н}\nu_\text{н}=0.
\end{aligned} \qquad (36.3)$$

The full flying range L is composed of range l_H prior to the initial point (for which we take point t_2) and range l_{CB} of free flight:

$$L=l_\text{н}+l_\text{св}, \qquad (36.4)$$

where

$$\begin{aligned}
l_\text{н}&=R\delta,\\
l_\text{св}&=R\beta_C,\\
\operatorname{tg}\delta&=\frac{x_\text{н}}{R+y_\text{н}}.
\end{aligned}$$

Quantities δ, h_H and ϑ_H are connected with x_H, y_H and θ_H by relations

$$\left.\begin{aligned}
R+y_\text{н}&=(R+h_\text{н})\cos\delta,\\
x_\text{н}&=(R+h_\text{н})\sin\delta,\\
\vartheta_\text{н}&=\theta_\text{н}+\delta.
\end{aligned}\right\} \qquad (36.5)$$

Let us trace how the flying range depends on kinematic quantities v_H, y_H, x_H and θ_H at the initial point of the section of free flight. With a change in v_H there is changed only l_{CB} in formula (36.4). With a change in x_H or y_H, h_H is changed affecting l_{CB}, and also δ, on which depends both l_H and l_{CB} (in terms of ϑ_H); finally, with a change in θ_H, ϑ_H changes, and together with it l_{CB}. Therefore,

$$\frac{\partial L}{\partial v_\text{н}}=\frac{\partial l_\text{св}}{\partial v_\text{н}}; \qquad (36.6)$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial y_\text{н}}&=\frac{\partial l_\text{н}}{\partial y_\text{н}}+\frac{\partial l_\text{св}}{\partial y_\text{н}}=R\frac{\partial\delta}{\partial y_\text{н}}+\frac{\partial l_\text{св}}{\partial h_\text{н}}\frac{\partial h_\text{н}}{\partial y_\text{н}}+\frac{\partial l_\text{св}}{\partial\vartheta_\text{н}}\frac{\partial\vartheta_\text{н}}{\partial y_\text{н}},\\
\frac{\partial L}{\partial x_\text{н}}&=\frac{\partial l_\text{н}}{\partial x_\text{н}}+\frac{\partial l_\text{св}}{\partial x_\text{н}}=R\frac{\partial\delta}{\partial x_\text{н}}+\frac{\partial l_\text{св}}{\partial h_\text{н}}\frac{\partial h_\text{н}}{\partial x_\text{н}}+\frac{\partial l_\text{св}}{\partial\vartheta_\text{н}}\frac{\partial\vartheta_\text{н}}{\partial x_\text{н}};
\end{aligned}\right\} \qquad (36.7)$$

$$\frac{\partial L}{\partial\theta_\text{н}}=\frac{\partial l_\text{св}}{\partial\theta_\text{н}}=\frac{\partial l_\text{св}}{\partial\vartheta_\text{н}}. \qquad (36.8)$$

For the calculation of partial derivatives $\frac{\partial h_\text{н}}{\partial y_\text{н}}$, $\frac{\partial h_\text{н}}{\partial x_\text{н}}$, $\frac{\partial\delta}{\partial y_\text{н}}$ and $\frac{\partial\delta}{\partial x_\text{н}}$ we will differentiate relations (36.5):

$$\begin{aligned}
dy_\text{н}&=dh_\text{н}\cos\delta-(R+h_\text{н})\sin\delta\,d\delta,\\
dx_\text{н}&=dh_\text{н}\sin\delta+(R+h_\text{н})\cos\delta\,d\delta.
\end{aligned}$$

whence

$$dh_{\text{н}} = dy_{\text{н}}\cos\delta + dx_{\text{н}}\sin\delta,$$
$$d\delta = -\frac{\sin\delta}{R+h_{\text{н}}}dy_{\text{н}} + \frac{\cos\delta}{R+h_{\text{н}}}dx_{\text{н}}.$$

Consequently,

$$\left.\begin{aligned}\frac{\partial h_{\text{н}}}{\partial y_{\text{н}}} &= \cos\delta, \quad \frac{\partial h_{\text{н}}}{\partial x_{\text{н}}} = \sin\delta,\\ \frac{\partial\delta}{\partial y_{\text{н}}} &= -\frac{\sin\delta}{R+h_{\text{н}}}, \quad \frac{\partial\delta}{\partial x_{\text{н}}} = \frac{\cos\delta}{R+h_{\text{н}}}.\end{aligned}\right| \tag{36.9}$$

On the basis of the last of formulas (36.5)

$$\frac{\partial\theta_{\text{н}}}{\partial y_{\text{н}}} = \frac{\partial\delta}{\partial y_{\text{н}}}, \quad \frac{\partial\theta_{\text{н}}}{\partial x_{\text{н}}} = \frac{\partial\delta}{\partial x_{\text{н}}}. \tag{36.10}$$

Substituting expressions (36.9) and (36.10) into formulas (36.7), we will obtain

$$\left.\begin{aligned}\frac{\partial L}{\partial y_{\text{н}}} &= \cos\delta\frac{\partial L_{\text{св}}}{\partial y_{\text{н}}} - \frac{\sin\delta}{R+h_{\text{н}}}\left(R + \frac{\partial L_{\text{св}}}{\partial\theta_{\text{н}}}\right),\\ \frac{\partial L}{\partial x_{\text{н}}} &= \frac{\cos\delta}{R+h_{\text{н}}}\left(R + \frac{\partial L_{\text{св}}}{\partial\theta_{\text{н}}}\right) + \sin\delta\frac{\partial L_{\text{св}}}{\partial h_{\text{н}}}.\end{aligned}\right| \tag{36.11}$$

Thus the connection between small deviations of speed, coordinates and the angle of inclination of the tangent in the beginning of the section of free flight and the deviation of the flying range can be expressed by the formula

$$\Delta L = \frac{\partial L}{\partial v_{\text{н}}}\Delta v_{\text{н}} + \frac{\partial L}{\partial y_{\text{н}}}\Delta y_{\text{н}} + \frac{\partial L}{\partial x_{\text{н}}}\Delta x_{\text{н}} + \frac{\partial L}{\partial\theta_{\text{н}}}\Delta\theta_{\text{н}}. \tag{36.12}$$

where coefficients with deviations Δv_H, Δy_H, Δx_H, $\Delta\theta_H$ are determined by the formulas (36.1)-(36.3), (36.6), (36.8) and (36.11), and the actual deviations by formulas of the preceding paragraphs.

Let us turn to concrete methods of the turning off of the engine.

With turning off at the fixed moment of time, using formulas (34.2), (35.8) and (35.10), we obtain

$$\begin{aligned}\Delta L = {} & \frac{\partial L}{\partial v_{\text{н}}}\left(\dot v_1\Delta t_{\text{н}} + \sum z_{1k}\Delta\lambda_k + \Delta v_{12}\right) + \\ & + \frac{\partial L}{\partial y_{\text{н}}}\left(\dot y_1\Delta t_{\text{н}} + \sum z_{2k}\Delta\lambda_k\right) + \\ & + \frac{\partial L}{\partial x_{\text{н}}}\left(\dot x_1\Delta t_{\text{н}} + \sum z_{3k}\Delta\lambda_k\right) + \frac{\partial L}{\partial\theta_{\text{н}}}\Delta\theta_{\text{н}}.\end{aligned}$$

Let us introduce designations:

$$\frac{\partial L}{\partial v_{\text{н}}}\dot v_1 + \frac{\partial L}{\partial y_{\text{н}}}\dot y_1 + \frac{\partial L}{\partial x_{\text{н}}}\dot x_1 = \dot L_1. \tag{36.13}$$

$$\frac{\partial L}{\partial v_{\text{н}}}\Delta v_{12} = \Delta L_{12}. \tag{36.14}$$

$$\frac{\partial L}{\partial\theta_{\text{н}}}\Delta\theta_{\text{н}} = \Delta L_{\theta}. \tag{36.15}$$

$$\frac{\partial L}{\partial v_{\text{н}}}z_{1k}^{(n)} + \frac{\partial L}{\partial y_{\text{н}}}z_{2k}^{(n)} + \frac{\partial L}{\partial x_{\text{н}}}z_{3k}^{(n)} = z_{4k}^{(n)}. \tag{36.16}$$

where the superscript (n) designates the quantity of primes, i.e., indicates the method of the turning off.

With these designations

$$\Delta L = \dot L_1\Delta t_{\text{н}} + \sum z_{4k}\Delta\lambda_k + \Delta L_{12} + \Delta L_{\theta}. \tag{36.17}$$

Let us note that this formula remains correct if the deviation of the time of turning off the engine $\Delta t_{И}$ is caused not by errors of measurement but by any other causes.

Passing to the turning off with respect to speed, we use formulas (34.3), (35.8) and (35.10):

$$\Delta L' = \frac{\partial L}{\partial v_к}(\Delta v_к + \Delta v_{12}) + \frac{\partial L}{\partial y_к}\left(\frac{\dot{y}_1}{\dot{v}_1}\Delta v_к + \sum z'_{2k}\Delta\lambda_k\right) +$$
$$+ \frac{\partial L}{\partial x_к}\left(\frac{\dot{x}_1}{\dot{v}_1}\Delta v_к + \sum z'_{3k}\Delta\lambda_k\right) + \frac{\partial L}{\partial \theta_к}\Delta\theta_к.$$

or, with the again introduced designations,

$$\Delta L' = \frac{\dot{L}_1}{\dot{v}_1}\Delta v_к + \sum z'_{4k}\Delta\lambda_k + \Delta L_{12} + \Delta L_{\theta}. \qquad (36.18)$$

For the turning off from the integrator on the basis of formulas (34.18), (35.8) and (35.10) we obtain:

$$\Delta L'' = \frac{\partial L}{\partial v_к}\left(\frac{\Delta v_{зи} + \Delta v_{зм}}{\dot{v}_1 + g_1 \sin\varphi_1}\dot{v}_1 + \sum z''_{1k}\Delta\lambda_k + \Delta v_{12}\right) +$$
$$+ \frac{\partial L}{\partial y_к}\left(\frac{\Delta v_{зи} + \Delta v_{зм}}{\dot{v}_1 + g_1 \sin\varphi_1}\dot{y}_1 + \sum z''_{2k}\Delta\lambda_k\right) +$$
$$+ \frac{\partial L}{\partial x_к}\left(\frac{\Delta v_{зи} + \Delta v_{зм}}{\dot{v}_1 + g_1 \sin\varphi_1}\dot{x}_1 + \sum z''_{3k}\Delta\lambda_k\right) + \frac{\partial L}{\partial \theta_к}\Delta\theta_к.$$

In designations (36.13)-(36.16)

$$\Delta L'' = \frac{\dot{L}_1}{\dot{v}_1 + g_1 \sin\varphi_1}(\Delta v_{зи} + \Delta v_{зм}) + \sum z''_{4k}\Delta\lambda_k + \Delta L_{12} + \Delta L_{\theta}. \qquad (36.19)$$

Let us analyze formula (36.17)-(36.19). The first members in the right sides of these formulas depend on $\Delta t_{И}$, $\Delta v_{И}$, $\Delta v_{SИ}$, i.e., they appear due to instrumental errors of instruments controlling the turning off of the engine. They can be decreased owing to design improvement of these instruments but cannot be completely suppressed, since ideally exact instruments do not exist. Members of the form $\sum z_{4k}^{(n)}\Delta\lambda_k$ constitute methodical errors of instruments of the turning off of the engine. They appear due to the fact that these instruments control parameters not connected directly with flying range, – time, speed or apparent speed. If certain source, for example the deviation $\Delta\lambda_k$ of parameter λ_k, caused a change in trajectory, then the flying range will be changed by the magnitude $z_{4k}^{(n)}\Delta\lambda_k$ even under the condition that the controlled parameter at the time of the turning off exactly is equal to the assigned value. Methodical errors can be considerably lowered and almost even suppressed (see § 37) as a result of the improvement of the principle of operation of instruments of range control. In particular, calculations show that the application of the integrator instead of the timing mechanism reduces methodical errors ten times, and replacement of the integrator by the turning off with respect to true speed additionally gives approximately a triple reduction in methodical errors.

Also methodical error is the last member in formulas (36.17)-(36.19). It appears because with not one of the examined methods of turning off is there considered the influence of the angle of inclination of the tangent on range at the time of the turning off. It can be eliminated by considering this influence with turning off of the engine. But it is possible to proceed another way, by selecting the form of the trajectory in such a manner that the influence of deviations in the angle of inclination on distance is reduced to zero. With this it would be insufficient to reduce to zero the last member in formulas (36.17)-(36.19), seeking fulfillment of the equality $\frac{\partial L}{\partial \theta_H} = 0$. It is necessary to consider the influence on range of not only the final angle of inclination of the tangent but also of the change in the angle of inclination of the tangent during the period of the entire controlled flight. For us this influence is characterized by members $x_{\theta}^{(n)}\Delta\lambda_{s}+x_{\theta}^{(n)}\Delta\lambda_{\tau}$, and in general they should be replaced (in sum with ΔL_θ) by the variation in full range depending upon the variation of function $\theta(t)$. Thus the problem of the removal of range dispersion appearing due to the deviation of the angle of inclination of the tangent is a variational problem.

Let us note that the variation of full range(and in our case, the members $x_{\theta}^{(n)}\Delta\lambda_{s}+x_{\theta}^{(n)}\Delta\lambda_{\tau}$) depends on the method of the turning off of the engine. Consequently, the solution to the variational problem will also depend on it. These questions are examined in Chapter XI in greater detail.

Irrespective of the method of turning off, into the range error there is introduced the quantity ΔL_{12}, which is the deviation due to possible scattering of forces having an effect on the rocket after the command is fed for final turning off of the engine, i.e., after complete cessation of control. In a number of these forces there appear, first of all, thrust, then drag, and also forces connected with the design of the rocket (for example, if from the rocket parts are rejected, then the pulse transmitted to these parts is sent to the rocket in the opposite direction). The methods of the decrease in quantity ΔL_{12} were discussed above in connection with possibilities of a decrease in Δv_{12}.

§ 37. Methods of Decrease in Dispersion

It was established above that range dispersion depends both on instrumental and methodical errors of the control system, mainly from errors of instruments turning off the engine. Let us consider in broad terms possible ways of reducing these errors.

Let us start from integrators of axial G-forces. As was mentioned above, the methodical error in range, obtained with the turning off from the integrator, is three times higher than the error with the maintaining a constant speed of flight of the rocket during the turning off. In other words, a greater part of the error in turning off from the integrator appears owing to the deviation of speed at the time of the turning off. Formula (34.14) shows that a deviation in speed appears due to the following of three factors: instrumental error of the integrator, deviation of the axis of the rocket and deviation of the time of turning off, where in numerical examples it is easy to check that the deviation of the time of turning off plays in this case a decisive role. This suggests to combine the integrator with the timing mechanism, i.e., to turn off the engine when a certain function from the apparent v_s and time t

$$v_t = v_t(v_s, t) \tag{37.1}$$

reaches the assigned value

$$v_t(v_s, t) = v_{tH}. \tag{37.2}$$

Such a procedure is called introduction into the integrator by time compensation.

For the mean trajectory the apparent speed is uniquely connected with the time of flight, and, consequently, the mean values of both the apparent speed at the time of the turning off and of the actual time of the turning off are uniquely determined by formula (37.2). But for every actual trajectory equality (37.2) will be fulfilled with other values of apparent and time distinguished from the mean by Δv_{s1} and Δt_1 respectively. In virtue of equality (37.2) we have

$$\frac{\partial v_t}{\partial v_s}\Delta v_{s1}+\frac{\partial v_t}{\partial t}\Delta t_1=\Delta v_{t\text{и}}$$

correct to magnitudes of higher order of smallness. Here $\Delta v_{t\text{И}}$ is the instrumental error of the instrument producing the quantity v_t.

Hence

$$\Delta v_{s1}=k_1\Delta t_1+\frac{1}{\frac{\partial v_t}{\partial v_s}}\Delta v_{t\text{и}},$$

where

$$k_1=-\frac{\partial v_t}{\partial t}\Big/\frac{\partial v_t}{\partial v_s}. \qquad (37.3)$$

Coefficient k_1 is called the coefficient of compensation of the integrator. Our problem will be selected by this coefficient in such a way as to reduce as far as possible the range dispersion. Substituting in formula (34.14) Δv_{s1} instead of Δv_{s}, we obtain the following connection between the deviation of speed at the time of the turning off and the deviation of the turning off:

$$\Delta v_1=(k_1-g_1\sin\varphi_1)\Delta t_1+\Delta v_{sp}+\frac{1}{\frac{\partial v_t}{\partial v_s}}\Delta v_{t\text{и}}.$$

Using the first formula (34.1), we have:

$$(k_1-g_1\sin\varphi_1)\Delta t_1+\Delta v_{sp}+\frac{1}{\frac{\partial v_t}{\partial v_s}}\Delta v_{t\text{и}}=\dot v_1\Delta t_1+\sum z_{1k}\Delta\lambda_k.$$

$$\begin{aligned}\Delta t_1={}&\frac{1}{\dot v_1+g_1\sin\varphi_1-k_1}\Delta v_{sp}+\\&+\frac{1}{\frac{\partial v_t}{\partial v_s}(\dot v_1+g_1\sin\varphi_1-k_1)}\Delta v_{t\text{и}}-\\&-\sum\frac{z_{1k}}{\dot v_1+g_1\sin\varphi_1-k_1}\Delta\lambda_k.\end{aligned} \qquad (37.4)$$

Formulas (34.1) give with this

$$\left.\begin{aligned}\Delta x_1&=\frac{\dot x_1}{\dot v_1+g_1\sin\varphi_1-k_1}\Delta v_{sp}+\frac{\dot x_1}{\frac{\partial v_t}{\partial v_s}(\dot v_1+g_1\sin\varphi_1-k_1)}\Delta v_{t\text{и}}+\sum\left(z_{2k}-\frac{\dot x_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1-k_1}\right)\Delta\lambda_k.\\ \Delta y_1&=\frac{\dot y_1}{\dot v_1+g_1\sin\varphi_1-k_1}\Delta v_{sp}+\frac{\dot y_1}{\frac{\partial v_t}{\partial v_s}(\dot v_1+g_1\sin\varphi_1-k_1)}\Delta v_{t\text{и}}+\sum\left(z_{3k}-\frac{\dot y_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1-k_1}\right)\Delta\lambda_k.\\ \Delta z_1&=\frac{\dot z_1}{\dot v_1+g_1\sin\varphi_1-k_1}\Delta v_{sp}+\frac{\dot z_1}{\frac{\partial v_t}{\partial v_s}(\dot v_1+g_1\sin\varphi_1-k_1)}\Delta v_{t\text{и}}+\sum\left(z_{4k}-\frac{\dot z_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1-k_1}\right)\Delta\lambda_k.\end{aligned}\right\} \qquad (37.5)$$

Using these formulas and also relations (35.8)-(35.10) and (36.12), we obtain in designations (36.13)-(36.16)

$$\Delta L=\frac{L_1}{\dot{v}_1+g_1\sin\varphi_1-k_1}\Delta v_{rn}+$$

$$+\frac{L_1}{\frac{\partial v_1}{\partial v_2}(\dot{v}_1+g_1\sin\varphi_1-k_1)}\Delta v_{1n}+$$

$$+\sum\left(z_{4k}-\frac{L_1 z_{1k}}{\dot{v}_1+g_1\sin\varphi_1-k_1}\right)\Delta\lambda_k+\Delta L_{13}+\Delta L_{\varphi}.$$

Let us examine the part of the deviation ΔL induced only be random deviations of parameters λ_k:

$$\Delta L_{\lambda}=\sum\left(z_{4k}-\frac{L_1 z_{1k}}{\dot{v}_1+g_1\sin\varphi_1-k_1}\right)\Delta\lambda_k.$$

Let us assume that λ_k are independent values subordinated to the normal law of distribution with standard deviations $\sigma\lambda_k$. Then on the basis of formula (31.4)

$$(\sigma L_{\lambda})^2=\sum\left(z_{4k}-\frac{L_1 z_{1k}}{\dot{v}_1+g_1\sin\varphi_1-k_1}\right)^2(\sigma\lambda_k)^2.$$

We take for the independent variable

$$q=\frac{L_1}{\dot{v}_1+g_1\sin\varphi_1-k_1} \qquad (37.6)$$

and find the minimum with respect to q of quantity

$$(\sigma L_{\lambda})^2=\sum(z_{4k}-qz_{1k})^2(\sigma\lambda_k)^2. \qquad (37.7)$$

For this let us note that the derivative

$$\frac{\partial(\sigma L_{\lambda})^2}{\partial q}=-2\sum(z_{4k}-qz_{1k})z_{1k}(\sigma\lambda_k)^2=$$

$$=-2\sum z_{4k}z_{1k}(\sigma\lambda_k)^2+2q\sum z_{1k}^2(\sigma\lambda_k)^2$$

turns into zero when

$$q=\frac{\sum z_{4k}z_{1k}(\sigma\lambda_k)^2}{\sum(z_{1k}\sigma\lambda_k)^2}. \qquad (37.8)$$

With this the standard deviation in range due to the deviation of only parameters λ_k will be minimum, since the second derivative

$$\frac{\partial^2(\sigma L_{\lambda})^2}{\partial q^2}=2\sum z_{1k}^2(\sigma\lambda_k)^2.$$

is positive. From (37.6) and (37.8) we find value of k_1, which will be called optimum for the time t_1:

$$k_1=\dot{v}_1+g_1\sin\varphi_1-\frac{L_1}{q}=\dot{v}_1+g_1\sin\varphi_1-\frac{L_1\sum(z_{1k}\sigma\lambda_k)^2}{\sum(z_{1k}\sigma\lambda_k)(z_{4k}\sigma\lambda_k)}. \qquad (37.9)$$

This formula determines the optimum coefficient of compensation of integrator k_1 as a function of the time t_1. From (37.3) we obtain that function v_t should satisfy the partial differential equation

$$k_1 \frac{\partial v_t}{\partial v_s} + \frac{\partial v_t}{\partial t} = 0. \tag{37.10}$$

The simplest solution of such an equation will be the function

$$v_t = v_s - \int_{t_0}^{t} k_1\, dt. \tag{37.11}$$

Time t_0 is selected arbitrarily but in such a manner so that it does not exceed the time of work of the engine with firing at minimum range. If a combination of the integrator with the timing mechanism producing quantity (37.11) is constructively difficult to realize, as a function of v_t it is possible to take

$$v_t = v_s - k_1 t. \tag{37.12}$$

For such a function equation (37.10) will be satisfied only during the time of the turning off equal to t_1. Consequently, at the assigned distance there should be tuned not only the quantity v_t, according to which will be turned off the engine, but also quantity k_1, the coefficient of compensation.

This method of the decrease in dispersion can give good results under the condition that the errors $\Delta v_{s\varphi}$, ΔL_{12}, ΔL_θ are small, and chiefly if coefficients z_{4k} and z_{1k} preserve an approximately constant ratio for different parameters of λ_k, expecially for those which give greater values of the product $z_{4k}\sigma\lambda_k$, i.e., greatly affect the range. This can be seen from formula (37.7).

Let us dwell in broad terms on other methods of reducing dispersion with the use of the integrator. Let us consider the integrator stabilized in space, i.e., with a constant inclination of the axis of sensitivity ($\varphi^* = \text{const}$). For such an integrator formula (34.9) will take the form

$$v_{s1} = v_1 \cos\alpha_1^* + \sin\varphi^* \int_0^{t_1} g\, dt = v_1 \cos(\varphi^* - \theta_1) + \sin\varphi^* \int_0^{t_1} g\, dt.$$

If the command is fed at the assigned value v_{s1}, then

$$\Delta v_1 \cos(\varphi^* - \theta_1) + v_1 \sin(\varphi^* - \theta_1)\,\Delta\theta_1 + g_1 \sin\varphi^*\,\Delta t_1 = 0. \tag{37.13}$$

We substitute here the expression for Δv_1 from formulas (34.1):

$$\begin{aligned}&\left(\dot{v}_1 \Delta t_1 + \sum z_{1k}\,\Delta\lambda_k\right)\cos(\varphi^* - \theta_1) + \\ &\qquad + v_1 \sin(\varphi^* - \theta_1)\,\Delta\theta_1 + g_1 \sin\varphi^*\,\Delta t_1 = 0.\end{aligned}$$

whence

$$\Delta t_1 = -\frac{v_1 \sin(\varphi^* - \theta_1)\,\Delta\theta_1 + \cos(\varphi^* - \theta_1)\sum z_{1k}\,\Delta\lambda_k}{\dot{v}_1 \cos(\varphi^* - \theta_1) + g_1 \sin\varphi^*}.$$

If we substitute this expression for Δt_1 instead of $\Delta t_{\text{И}}$ into formula (36.17), then we obtain, considering expression (36.15):

$$\Delta L = \sum\left(z_{4k} - \frac{L_1 \cos(\varphi^* - \theta_1)\, z_{1k}}{\dot{v}_1 \cos(\varphi^* - \theta_1) + g_1 \sin\varphi^*}\right)\Delta\lambda_k + \left(\frac{\partial L}{\partial\theta_{\text{к}}} - \frac{L_1 v_1 \sin(\varphi^* - \theta_1)}{\dot{v}_1 \cos(\varphi^* - \theta_1) + g_1 \sin\varphi^*}\right)\Delta\theta_1 + \Delta L_{12}.$$

Hence it is clear that with the proper selection of inclination of the axis of sensitivity φ^* it is possible to compensate part of the influence of the final angle of inclination of the tangent on the flying range, turning in zero the coefficient with $\Delta\theta_1$, in the obtained expression for ΔL. Really, equating this coefficient to zero, we obtain

$$\frac{\partial L}{\partial \theta_k} - \frac{L_1 v_1 \sin(\varphi^* - \theta_1)}{\dot{v}_1 \cos(\varphi^* - \theta_1) + g_1 \sin\varphi^*} = 0.$$

This equation is easily solved with respect to tg φ^*:

$$\operatorname{tg}\varphi^* = \frac{L_1 v_1 \sin\theta_1 + \frac{\partial L}{\partial \theta_k}\dot{v}_1 \cos\theta_1}{L_1 v_1 \cos\theta_1 - \frac{\partial L}{\partial \theta_k}(\dot{v}_1 \sin\theta_1 + g_1)}.$$

However, as was already discussed at the end of the preceding paragraph, destruction of the member with $\Delta\theta_1$ in the formula for ΔL does not completely remove the influence of deviations of the angle of inclination of the tangent on the range error. A more improved approach to the solution of this problem would be to find the dependence σL on angle φ^* and then to determine the value of φ^* delivering the minimum σL, as this was done above with the determination of the optimum coefficient of compensation.

However, the formulas derived by us are insufficient for solution of the problem in such a formulation (those readers which managed to advance forward, adding to the system (33.1) the equation for $\frac{d\theta}{dt}$, are in the best position). It is possible to go even further: to introduce into integrator, the axis of sensitivity of which is inclined to the horizon at a constant angle φ^* (or to the longitudinal axis of the rocket at the constant angle γ^*), the time compensation with coefficient k_1, to examine the dependence σL on two parameters φ^* (or γ^*) and k_1, and to find the minimum of this function of two variables. This will allow reducing dispersion to even smaller values than with the use of only time compensation or only the setting of the integrator at an arbitrary angle.

It is possible to try to obtain further improvement by using a double integrator, which in combination with the timing mechanism can reduce to nought range errors due to deviations in speed, coordinates and slope of the tangent at the time of the turning off of the engine. This is carried out by means of proper selection of the coefficient of compensation and directions of sensitivity with the first and second integration. Range dispersion will remain only owing to instrumental errors of the first and second integration, the scattering of forces effective after the command for full turning off of the engine, and the disturbances obtained by the rocket during free flight.

Being distracted with what instrument will turn off the engine, we will obtain the equation which such an instrument should operate in order to reduce to a minimum the methodical errors. We will proceed from equations (35.13), (35.15) and (36.12), taking for the initial point of free flight the point t_2:

$$\Delta L = \frac{\partial L}{\partial v_k}(\Delta v_k + \Delta v_{kc}) + \frac{\partial L}{\partial y_k}\Delta y_k + \frac{\partial L}{\partial x_k}\Delta x_k + \frac{\partial L}{\partial \theta_k}\Delta\theta_k. \tag{37.14}$$

Since t_K is close to t_2, one can assume that

$$\Delta\theta_k = \Delta\theta_2 = \Delta\theta_{\pi}.$$

The instrument, producing the quantity

$$L' = \frac{\partial L}{\partial v_k}v + \frac{\partial L}{\partial y_k}y + \frac{\partial L}{\partial x_k}x + \frac{\partial L}{\partial \theta_k}\theta \tag{37.15}$$

and sending the command to turn off the engine at the time when this quantity attains the assigned value

$$L'_{\text{расч}}=\frac{\partial L}{\partial v_{\text{н}}}v_{\text{расч}}+\frac{\partial L}{\partial y_{\text{н}}}y_{\text{расч}}+\frac{\partial L}{\partial x_{\text{н}}}x_{\text{расч}}+\frac{\partial L}{\partial \theta_{\text{н}}}\theta_{\text{расч}}.$$

satisfies the mentioned requirement. Actually, from (37.15)

$$\Delta L'_{\text{и}}=\frac{\partial L}{\partial v_{\text{н}}}(\Delta v_{\text{к}}+\Delta v_{\text{и}})+\frac{\partial L}{\partial y_{\text{н}}}(\Delta y_{\text{к}}+\Delta y_{\text{и}})+ \\ +\frac{\partial L}{\partial x_{\text{н}}}(\Delta x_{\text{к}}+\Delta x_{\text{и}})+\frac{\partial L}{\partial \theta_{\text{н}}}(\Delta\theta_{\text{к}}+\Delta\theta_{\text{и}}). \qquad (37.16)$$

where $\Delta L'_{\text{Д}}$ is the error in development of the command; $\Delta v_{\text{К}}$ — deviation of actual speed from calculated at the time of the turning off; $\Delta v_{\text{И}}$ — deviation of measured speed from the actual; $\Delta y_{\text{К}}$, $\Delta x_{\text{К}}$, $\Delta\theta_{\text{К}}$ $\Delta y_{\text{И}}$, $\Delta x_{\text{И}}$, $\Delta\vartheta_{\text{И}}$ are analogous quantities for coordinates and the angle of inclination of the tangent at the time of the turning off.

Comparing (37.14) and (37.16), we obtain

$$\Delta L=\Delta L'_{\text{и}}+\frac{\partial L}{\partial v_{\text{н}}}(-\Delta v_{\text{и}}+\Delta v_{\text{и}2})-\frac{\partial L}{\partial y_{\text{н}}}\Delta y_{\text{и}}- \\ -\frac{\partial L}{\partial x_{\text{н}}}\Delta x_{\text{и}}-\frac{\partial L}{\partial \theta_{\text{н}}}\Delta\theta_{\text{и}}. \qquad (37.17)$$

It is clear that methodical errors decrease with a similar method of turning off. In reality small methodical errors remain due to the inaccuracy of expression (37.14), in which members of second order about $\Delta v_{\text{К}}$, $\Delta y_{\text{К}}$, $\Delta x_{\text{К}}$, $\Delta\vartheta_{\text{К}}$ are rejected.

In formula (37.15) quantity L' depends on four kinematic parameters: v, y, x and ϑ. But the same result can be obtained being limited by measurement of only two parameters: projections of speed on a direction which form with the calculated direction of the tangent to the trajectory angle ω, determined from the relation

$$\operatorname{tg}\omega=\frac{\dfrac{\partial L}{\partial \theta_{\text{н}}}}{v\dfrac{\partial L}{\partial v_{\text{н}}}}.$$

and projections of passed by the rocket, on a direction forming with the horizon of the point of launch the angle ψ, where

$$\operatorname{tg}\psi=\frac{\partial L}{\partial y_{\text{н}}}\Big/\frac{\partial L}{\partial x_{\text{н}}}.$$

Really, we will designate the first of these parameters v_{ω} and the second s_{ψ}. For v_{ω} the expression

$$v_{\omega}=v\cos\omega.$$

is correct. Let us calculate Δv_{ω} correct to the linear members, noticing that with a change in the actual direction of the tangent to the trajectory $\Delta\omega = -\Delta\theta$:

$$\Delta v_{\omega}=\Delta v\cos\omega-v\sin\omega\,\Delta\omega=\Delta v\cos\omega+v\sin\omega\,\Delta\theta.$$

Hence

$$\frac{1}{\cos\omega}\frac{\partial L}{\partial v_{\text{н}}}\Delta v_{\omega}=\frac{\partial L}{\partial v_{\text{н}}}\Delta v+v\frac{\partial L}{\partial v_{\text{н}}}\operatorname{tg}\omega\,\Delta\theta= \\ =\frac{\partial L}{\partial v_{\text{н}}}\Delta v+\frac{\partial L}{\partial \theta_{\text{н}}}\Delta\theta. \qquad (37.18)$$

Further,

$$s_{\psi}=x\cos\psi+y\sin\psi.$$

consequently,

$$\Delta s_\psi = \Delta x \cos\psi + \Delta y \sin\psi$$

and

$$\frac{1}{\cos\psi}\frac{\partial L}{\partial x_\text{н}}\Delta s_\psi = \frac{\partial L}{\partial x_\text{н}}\Delta x + \frac{\partial L}{\partial x_\text{н}}\operatorname{tg}\psi\,\Delta y = \\ = \frac{\partial L}{\partial x_\text{н}}\Delta x + \frac{\partial L}{\partial y_\text{н}}\Delta y. \qquad (37.19)$$

Thus, if the instrument produces the quantity

$$L'' = \frac{1}{\cos\omega}\frac{\partial L}{\partial v_\text{н}}v_\omega + \frac{1}{\cos\psi}\frac{\partial L}{\partial x_\text{н}}s_\psi$$

and sends a command to turn off the engine at that moment when this quantity reaches the computed value, then for the moment of feeding of this command the following relation between actual deviations $\Delta v_{\omega\text{к}}$ and $\Delta s_{\psi\text{к}}$ of quantities v_ω and s_ψ and errors of measurements $\Delta L''_\text{и}$, $\Delta v_{\omega\text{и}}$, and $\Delta s_{\psi\text{и}}$ is correct:

$$\Delta L''_\text{и} = \frac{1}{\cos\omega}\frac{\partial L}{\partial v_\text{н}}(\Delta v_{\omega\text{к}} + \Delta v_{\omega\text{и}}) + \frac{1}{\cos\psi}\frac{\partial L}{\partial x_\text{н}}(\Delta s_{\psi\text{к}} + \Delta s_{\psi\text{и}}).$$

Hence we obtain, taking into account formulas (37.14), (37.18) and (37.19):

$$\Delta L = \Delta L''_\text{и} - \frac{1}{\cos\omega}\frac{\partial L}{\partial v_\text{н}}\Delta v_{\omega\text{и}} - \frac{1}{\cos\psi}\frac{\partial L}{\partial x_\text{н}}\Delta s_{\psi\text{и}} + \frac{\partial L}{\partial v_\text{н}}\Delta v_{\text{кт}}. \qquad (37.20)$$

Thus just as in formula (37.17) there are absent here linear components of methodical errors induced by deviation of kinematic parameters at the end of the powered section. The range error depends only on errors of measurements and calculation of quantity L" and on causes effective after feeding the command for turning off the engine.

§ 38. Lateral Dispersion

Lateral dispersion is determined mainly by the following factors: errors in aiming in direction, deviations of coordinate z and lateral speed v_z from their computed values, and disturbances having an effect on the free-flight section.

The error in aiming leads to the fact that quantities z and v_z, and also any other quantities controlled by the control system, for example, the angle of yaw ξ, the component of apparent speed in the direction of the z axis, etc., are measured not in the system of coordinates in which it is necessary. It is clear that the control system cannot correct this error. This pertains both to a completely autonomous system and to one which uses ground measurements for control over the direction of flight of the rocket. In the second case the control system can reveal that the rocket was inaccurately oriented before launch and therefore began to move not in that plane. By commands from earth this inaccuracy can be corrected, and the rocket will be brought into the assigned plane. But this assigned plane itself can have an inaccurate direction and can even be not a plane but a slightly distorted surface because of errors in installation of ground control means (antennas of radar, direction finders, etc.). Thus with ground control although errors of aiming can be, as a rule, decreased, they can not be completely eliminated. It is natural to consider these errors as instrumental errors.

The task of control by yawing motion in principle differs from range control of the flight by the nature of the commands passed. For range control it is required to determine and exactly maintain only one quantity: the moment of turning off of the engine. It is true that for the determination of this quantity, as we have seen, can require quite a lot of measuring means, and the measurements should be conducted continuously, at least at the end of the powered section. For the control of yawing motion there are also needed uninterrupted measurements of kinematic parameters, on which this motion depends. But also commands of control of yawing motion should be sent and carried out continuously, so that towards the moment of turning off the engine, whenever it comes, all the disturbances affecting lateral deviation of the impact point would be compensated by a corresponding operation of the control devices. The continuous character of control requires the application of specific methods of investigation and calculation which, as was already stated above, usually do not pertain to the number of ballistic methods and are not examined in this book. In ballistics only partial derivatives of coordinates of the impact point L and Z are determined with respect to kinematic parameters of the point of turning off the engine. Just as for range controls, in the first place derivatives $\frac{\partial L}{\partial v_{\text{к}}}$, $\frac{\partial L}{\partial y_{\text{к}}}$, $\frac{\partial L}{\partial x_{\text{к}}}$ and $\frac{\partial L}{\partial \vartheta_{\text{к}}}$, and for control of yawing motion it is necessary, first of all, to know $\frac{\partial Z}{\partial v_{z\text{к}}}$ and $\frac{\partial Z}{\partial z_{\text{к}}}$.

In the first approximation lateral deviation is expressed by the formula

$$\Delta Z = \frac{\partial Z}{\partial v_{z\text{к}}} \Delta v_{z\text{к}} + \frac{\partial Z}{\partial z_{\text{к}}} \Delta z_{\text{к}}.$$

where Δv_z and $\Delta z_{\text{к}}$ are deviations of the lateral component of speed and z coordinate from their computed values.

However, for a great flying ranges, when all derivatives start to increase greatly and the space curvature of the trajectory becomes considerable, it is necessary to consider and other derivatives: $\frac{\partial L}{\partial v_{z\text{к}}}$ and $\frac{\partial L}{\partial z_{\text{к}}}$ in the calculation of range dispersion $\frac{\partial Z}{\partial v_{\text{к}}}$, $\frac{\partial Z}{\partial x_{\text{к}}}$, $\frac{\partial Z}{\partial y_{\text{к}}}$, $\frac{\partial Z}{\partial \vartheta_{\text{к}}}$ during the calculation of dispersion in a lateral direction. Sometimes it is necessary to take into account the second partial derivatives.

In order to make calculations of range and direction dispersion less dependent on each other, sometimes we measure deviations ΔL and ΔZ in a system of coordinates distinguished from that in which thus far we calculated L and Z. The origin of this system of coordinates is at the calculation point of impact (in other words, at the point of the target), axis ΔL is directed along a tangent to that line about the surface of earth along which moves the point of impact with a change in time of the turning off of the engine and with constant direction of aiming. Axis ΔZ is directed in a horizontal (at point of the target) plane at a right angle to axis ΔL.

§ 39. Calculation of Dispersion

In the preceding paragraphs there were discussed certain theoretical foundations and methods of calculation of range dispersion. Let us dwell more specifically on the calculating side of the matter, since in practice calculations are frequently conducted with deviations from above-stated scheme.

Partial derivatives z_{ik}, as was already stated, are determined with help of system (33.15). However, besides the direct numerical integration of this system there exist other procedures of detecting its solutions, allowing in certain cases to reduce the quantity of calculations. The most important of such procedures is the use of the conjugate system of differential equations.

Since it is frequently necessary to operate with systems of more general form than that of system (33.15), we will examine the system of the n order:

$$\left.\begin{array}{l} \frac{dz_{1k}}{dt} = a_{11} z_{1k} + \ldots + a_{1n} z_{nk} + \beta_{1k}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dz_{nk}}{dt} = a_{n1} z_{1k} + \ldots + a_{nn} z_{nk} + \beta_{nk}. \end{array}\right\} \quad (39.1)$$

where α_{ij}, β_{ik}, i, j = 1, 2, ..., n; k = 1, 2, ..., m, are well-known functions of time. Actually equalities (39.1) determine not one system but m such systems with the same coefficients α_{ij} with unknown functions z_{jk} and changing from system to system of free terms β_{ik}. The second subscript for quantities z_{jk} and β_{ik} is the number of the system.

Let us introduce auxiliary differentiable functions of time $u_1, \ldots, u_n$ and find the time derivative of function $u_1 z_{1k} + \ldots + u_n z_{nk}$:

$$\frac{d}{dt}(u_1 z_{1k} + \ldots + u_n z_{nk}) = \\ = \frac{du_1}{dt} z_{1k} + \ldots + \frac{du_n}{dt} z_{nk} + u_1 \frac{dz_{1k}}{dt} + \ldots + u_n \frac{dz_{nk}}{dt}.$$

Instead of derivatives $\frac{dz_{ik}}{dt}$ we insert their expression from equations (39.1):

$$\frac{d}{dt}(u_1 z_{1k} + \ldots + u_n z_{nk}) = \frac{du_1}{dt} z_{1k} + \ldots + \frac{du_n}{dt} z_{nk} + \\ + u_1(\alpha_{11} z_{1k} + \ldots + \alpha_{1n} z_{nk} + \beta_{1k}) + \\ + \ldots + u_n(\alpha_{n1} z_{1k} + \ldots + \alpha_{nn} z_{nk} + \beta_{nk}).$$

We will now group members containing $z_{1k}, \ldots, z_{nk}$:

$$\frac{d}{dt}(u_1 z_{1k} + \ldots + u_n z_{nk}) = \\ = \left(\frac{du_1}{dt} + \alpha_{11} u_1 + \ldots + \alpha_{n1} u_n\right) z_{1k} + \ldots + \left(\frac{du_n}{dt} + \alpha_{1n} u_1 + \ldots \right. \\ \left. \ldots + \alpha_{nn} u_n\right) z_{nk} + u_1 \beta_{1k} + \ldots + u_n \beta_{nk}. \tag{39.2}$$

The obtained expression will be considerably simplified, if one were to require that functions u_1 ..., u_n satisfy the system of differential equations:

$$\left.\begin{array}{l} \frac{du_1}{dt} = -\alpha_{11} u_1 - \ldots - \alpha_{n1} u_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{du_n}{dt} = -\alpha_{1n} u_1 - \ldots - \alpha_{nn} u_n. \end{array}\right| \tag{39.3}$$

This linear uniform system of the n-th order is called a system conjugate to system (39.1), more accurately, any of systems (39.1) obtained with different values of k. The matrix of coefficients of the conjugate system is obtained from the matrix of coefficients of the initial system by transposition and change of sign of all the elements.

Subsequently we will consider that functions $u_1, \ldots, u_n$ will form a solution of system (39.3). Then equality (39.2) takes the form

$$\frac{d}{dt}(u_1 z_{1k} + \ldots + u_n z_{nk}) = u_1 \beta_{1k} + \ldots + u_n \beta_{nk}.$$

Integrating this equality term by term from t = 0 to $t = t_1$, we obtain

$$(u_1 z_{1k} + \ldots + u_n z_{nk})\Big|_0^{t_1} = \int_0^{t_1} (u_1 \beta_{1k} + \ldots + u_n \beta_{nk})\, dt. \tag{39.4}$$

We will consider that the particular solution of system (39.1) interests us with initial conditions when t = 0:

$$z_{1k} = \ldots = z_{nk} = 0 \tag{39.5}$$

(so it is, in particular, for system (33.15) and those similar to it). Then result of substitution of the value t = 0 into the left side of equality (39.4) turns into zero, and the equality takes the form

$$(u_1 z_{1k}+\ldots+u_n z_{nk})|_{t=t_1}=\int_0^{t_1}(u_1\beta_{1k}+\ldots+u_n\beta_{nk})\,dt. \tag{39.6}$$

It is correct if $u_1, \ldots, u_n$ is the arbitrary particular solution of system (39.3). Let us now take some of the particular solutions of this system, namely, the solution $u_{1i}, \ldots, u_{ni}$, satisfying when $t = t_1$ the initial conditions

$$\left.\begin{array}{l} u_{ji}=0 \text{ when } j\neq i, \\ u_{ji}=1 \text{ when } j=i. \end{array}\right\} \tag{39.7}$$

Actually formula (39.7) determine not one particular solution of system (39.3) but n such solutions corresponding to different values i = 1, 2, ..., n. With substitution of such a particular solution into relation (39.6) all components of the left side except one will turn into zero, and we will obtain

$$z_{ik}|_{t=t_1}=\int_0^{t_1}(u_{1i}\beta_{1k}+\ldots+u_{ni}\beta_{nk})\,dt. \tag{39.8}$$

Thus if there are found n particular solutions of system (39.3) under initial conditions (39.7) for i = 1, 2, ..., n, then quantities z_{ik} at any k can be found, no longer resorting to the numerical integration of differential equations, but only with the help of a considerably less laborious process of calculation of definite integrals in formulas (39.8) for i = 1, 2, ..., n; k = 1, 2, ..., m. If the order n of system (39.1) is less than the number of m variants of this system, then the total volume of calculations is reduced. Moreover, frequently it is not a necessity to determine the values of all quantities z_{ik} when $t = t_1$. It is sufficient to be limited to the calculation of their certain linear combination

$$(c_1 z_{1k}+\ldots+c_n z_{nk})|_{t=t_1}, \tag{39.9}$$

where $c_1, \ldots, c_n$ are certain coefficients not depending on k. Thus from formulas (34.4) it is clear that such linear combinations are quantities z'_{0k}, z'_{2k} and z'_{3k}. Formulas (34.17) show that the same property is possessed by quantities z''_{0k}, z''_{1k}, z''_{2k} and z''_{3k}. Finally, proceeding from formula (36.16), it can be concluded that the same is correct with respect to $z^{(n)}_{4k}$. For example,

$$\begin{aligned} z''_{4k} &= \frac{\partial L}{\partial v_n} z''_{1k}+\frac{\partial L}{\partial y_n} z''_{2k}+\frac{\partial L}{\partial x_n} z''_{3k}=\frac{\partial L}{\partial v_n}\cdot\frac{g_1\sin\varphi_1}{\dot v_1+g_1\sin\varphi_1} z_{1k}+ \\ &+\frac{\partial L}{\partial y_n}\left(z_{2k}-\frac{\dot y_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1}\right)+\frac{\partial L}{\partial x_n}\left(z_{3k}-\frac{\dot x_1 z_{1k}}{\dot v_1+g_1\sin\varphi_1}\right)= \\ &=\frac{1}{\dot v_1+g_1\sin\varphi_1}\left(g_1\sin\varphi_1\frac{\partial L}{\partial v_n}-\dot y_1\frac{\partial L}{\partial y_n}-\dot x_1\frac{\partial L}{\partial x_n}\right) z_{1k}+ \\ &+\frac{\partial L}{\partial y_n} z_{2k}+\frac{\partial L}{\partial x_n} z_{3k}. \end{aligned}$$

Linear combinations of (39.9) for k = 1, 2, ..., m can be calculated having determined preliminarily all z_{ik} by the formula (39.8). This will require integrating system (39.3) n times with initial conditions (39.7) for all values i and then calculating n x m integrals (39.8). But if one were to integrate system (39.3) with initial conditions when $t = t_1$

$$u_1=c_1, \ldots, u_n=c_n \tag{39.10}$$

and to designate by $\bar{u}_1, \ldots, \bar{u}_n$ the obtained particular solution, then from formula (39.6) it follows that

$$(c_1 z_{1k} + \ldots + c_n z_{nk})|_{t=t_1} = \int_0^{t_1} (\bar{u}_1 \beta_{1k} + \ldots + \bar{u}_n \beta_{nk})\, dt. \tag{39.11}$$

Thus the calculation of linear combinations (39.9) for all values k can be reduced only to a single integration of system (39.3) with initial conditions (39.10) and to the calculation of m integrals (39.11).

The method of conjugate systems is not deprived of deficiencies. First, it permits calculating by the described scheme of value z_{ik} (z'_{ik}, z''_{ik}) only with one value t equal to t_1. If these values are needed for several values of t, then the problem is immediately complicated and becomes comparable in laboriousness with direct integration of system (39.1). Secondly, integration of the conjugate system of (39.3), the initial conditions for which are assigned when $t = t_1$, should be conducted with a decrease in t from t_1 to zero. Therefore, it is impossible to conduct it in parallel with integration of the basic system of differential equations of motion of the rocket in a direction from $t = 0$ to $t = t_1$. If with manual integration this does not cause special difficulties, then with the use of electronic computers it is necessary either to calculate coefficients α_{ij} and β_{ik} in parallel with integration of equations of motion and to store them in the memory of the machine (this requires a great volume of memory) or to repeat integration of the equations in the opposite direction (from $t = t_1$ to $t = 0$). In parallel with reverse integration coefficients α_{ij} and β_{ik} are calculated, the conjugate system (or systems) is integrated, and integrals (39.8) or (39.11) are calculated.

Both methods — direct integration of system (39.1) and the use of the conjugate system (39.3) — require calculation of a great quantity of coefficients α_{ij} and β_{ik} by rather bulky formulas. Therefore, with machine reading for standardization of calculations and reduction of the volume of the program the influence of small perturbations on the trajectory is frequently investigated by the method of finite differences. Let us explain this method with an example. Let us assume that it is required to find the derivative z''_{4k} of the flying range by parameter λ_k with the turning off of the engine from the integrator. At first there is calculated the nominal trajectory and determined the nominal flying range L_0 and value of apparent speed v_{s0} at the time of the turning off of the engine. Then there is calculated the perturbed trajectory for which all the parameters except λ_k are assigned by their nominal values, but parameter λ_k is given a value distinguished from the nominal by the highest possible value $+\Delta\lambda_k$. On this perturbed trajectory the moment of the turning off of the engine is selected in such a manner so that the value of the apparent speed at this instant would coincide with the earlier found nominal value v_{s0}. The section of free flight is miscalculated, and the disturbed value L''_{+k} is determined. If for some reason there is confidence in the fact that the dependence of the flying range on parameter λ_k is linear with a change of the latter in the examined limits, then it is possible to be limited by this and consider that

$$z''_{4k} = \frac{L''_{+k} - L_0}{\Delta\lambda_k}. \tag{39.12}$$

However, more frequently there is miscalculated absolutely analogously the perturbed trajectory corresponding to the maximum negative deviation, $\Delta\lambda_k$ of parameter λ_k at nominal values of remaining parameters. If L''_{-k} is the corresponding flying range, then formula

$$x'_{\lambda k} = \frac{L''_{+k} - L''_{-k}}{2\Delta\lambda_k} \qquad (39.13)$$

gives more a exact derivative than the preceding one, and expression

$$\left| \frac{L''_{+k} + L''_{-k}}{2} - L_0 \right|$$

can serve for an appriasal of nonlinearity of the dependence of distance on parameter λ_k. If this nonlinearity is great, then linear formulas of type of formulas (36.17)-(36.19) do not completely reflect the dependence of deviations of the flying range on deviations of design and other parameters. However, in an overwhelming majority of cases with nonlinearity in formulas of type shown, it is possible not to consider.

Therefore, in an approximate determination of derivatives of a higher order with a use of finite differences it is possible not to discuss, although this is done rather simply. Let us note that it is especially convenient to use finite differences when calculations are made on the computer. On these machines it is easy to provide a reserve of accuracy of calculations sufficient enough that with the subtraction of close values L''_{+k} and L''_{-k} there is preserved the required number of true signs. The fact that the actual calculations by the method of finite differences are made by more monotypic formulas was already mentioned. It is obvious that with the help of finite differences it is possible to calculate derivatives of the type $\frac{\partial L}{\partial v_k}$, $\frac{\partial L}{\partial y_k}$, $\frac{\partial Z}{\partial z_k}$, especially in those cases when the section of free flight is calculated by a more complicated method than was accepted in § 36, for example, by numerical integration of equations of motion considering drag and the flatness of earth.

One of the methods useful for investigation of the dispersion of rockets in the case of both linear and nonlinear dependences of coordinates of the point of impact on the perturbing factors is the method for which there has been given a number of names: method of static tests, method of Monte-Carlo, or, finally, the method of mathematical firings, a term best of all reflecting the essence of the matter. In this method dispersion is estimated on the basis of results of calculation of several tens of perturbed trajectories. Calculation is produced by as complete equations of motion as possible, in which there are considered all the known perturbing factors or, at least, those of them whose calculation does not complicate excessively the integration of equations of motion. Perturbing factors are selected in such a manner so that they physically, or at least in a probabilistic meaning, are independent of each other. Values of these perturbing factors are assigned as independent random quantities for each of the calculated trajectories and for every factor. As the basis of the assignment of these values there are set the well-known or assumed laws of the distribution of perturbing factors. As a rule, this normal law with the mean value is zero with its standard deviation for every factor. With the output of random values either tables of random numbers (usually with manual count) are used or special random number transducers connected to a computer, or subprograms producing sequences of so-called pseudorandom numbers externally behaving as accidental with the defined law of distribution.

For each of the perturbed trajectories there are calculated not only kinematic characteristics but also values controlled by a control system, in particular, a range control system. The moment of the turning off of the engine is determined proceeding from the selected control equation, i.e., the relation between magnitudes measured by the range control system according to which this system determines the moment of supply of the command for turning off. Thus at our disposal there appears a set of a certain number N of perturbed trajectories, more or less exactly reproducing trajectories which can be realized during actual launches of rockets. For each of these trajectories coordinates of the point of impact L_i and Z_i are determined, where i is the number of the trajectory (i = 1, 2, ..., N), which then are processed as if they are results of real launchings. Let us give well-known formulas by which such treatment is conducted. Mean values of range and lateral deviation are calculated by the formulas

$$\bar{L} = \frac{1}{N}\sum_{i=1}^{N} L_i. \tag{39.14}$$

$$\bar{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i. \tag{39.15}$$

These values are also called coordinates of the center of clustering of points of impact. Further treatment is given to the deviation of points of impact from the center of clustering: range error

$$\Delta L_i = L_i - \bar{L}$$

and deviation in a lateral direction

$$\Delta Z_i = Z_i - \bar{Z}.$$

The standard range deviation is found by the formula

$$\sigma L = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} \Delta L_i^2}. \tag{39.16}$$

and the standard deviation in a lateral direction, by the formula

$$\sigma Z = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} \Delta Z_i^2}\,. \tag{39.17}$$

It is possible to manage without calculation of deviations from the center of clustering and use directly deviations from the calculation point of impact; then formulas for σL and σZ will take the form

$$\sigma L = \sqrt{\frac{1}{N-1}\left(\sum_{i=1}^{N} (L_i - L_0)^2 - N(\bar{L} - L_0)^2\right)}. \tag{39.18}$$

where L_0 is the nominal flying range, and

$$\sigma Z = \sqrt{\frac{1}{N-1}\left(\sum_{i=1}^{N} Z_i^2 - N\bar{Z}^2\right)}. \tag{39.19}$$

These formulas give the same values as those preceding but permit reducing the calculations, especially if L_0 is a round number, which is important during manual count.

The correlation moment between range errors and in a lateral direction is estimated by the formula

$$K_{LZ} = \frac{1}{N-1}\sum_{i=1}^{N} \Delta L_i \Delta Z_i. \tag{39.20}$$

or

$$K_{LZ} = \frac{1}{N-1}\left(\sum_{i=1}^{N} (L_i - L_0) Z_i - N(\bar{L} - L_0)\bar{Z}\right). \tag{39.21}$$

If that moment is not equal to zero, then the dispersion of points of impact is characterized by an ellipse whose axes are turned with respect to axes L and Z. The angle of rotation α can be found from the relation

$$\operatorname{tg} 2\alpha = \frac{2K_{LZ}}{(\sigma L)^2 - (\sigma Z)^2}. \tag{39.22}$$

This relation determines the angle 2α correct to π and, consequently, angle α correct to π/2. In other words, formula (39.22) leaves the possibility of selection of one of two mutually perpendicular directions. So that selection becomes unique, it is necessary to consider that when $K_{LZ} > 0$ the major axis of the ellipse of dispersion is disposed between positive directions of axes L and Z, and when $K_{LZ} < 0$ it is between the positive direction one of these axes and negative direction of the other. Usually the standard deviations σL and σZ are close in value, and therefore the denominator of formula (39.22) and, at the same time, the value of α are determined with low accuracy. In the limit when σL = σZ if with this $K_{LZ} = 0$, the ellipse of dispersion is turned into a circle and angle α, determining the direction of axes of the ellipse, in general, becomes indefinite.

Other parameters characterizing dispersion, namely, $\overline{L}$, $\overline{Z}$, σL and σZ, are determined by this method with an accuracy quite sufficient for practical purposes. Thus, for example, if these parameters are estimated by results of the calculation 50 perturbed trajectories, then $\overline{L}$ is determined with an error not exceeding 0.4σL, and σL with a relative error of not more than 30%. In the same relation errors of the determination of $\overline{Z}$ and $\sigma\overline{Z}$ occur towards value of σZ. Usually errors of the determination of parameters of dispersion are found to be considerably less than the limits shown here. The cause of the appearance of these errors is clear; the calculation of perturbed trajectories is produced with the use of random numbers, and therefore the characteristics of dispersion of these trajectories are themselves random variables subject to scattering. It is possible to increase the accuracy of determination of these characteristics by increasing the number of trajectories, but the accuracy increases, more correctly the errors decrease, only proportionally to the square root of the number of perturbed trajectories, so that it is possible to determine σL or σZ with a guaranteed error of not over 10% from results of a calculation of about 500 trajectories. Therefore, as already was stated in the beginning, we usually put up with comparatively the highest possible error in the calculation of σL and σZ but are limited by the calculation of several tens of trajectories.

Quantities $\overline{L}$, $\overline{Z}$, σL, σZ and K_{LZ} completely characterize the dispersion of points of impact if this dispersion obeys the two-dimensional normal law of distribution. As a rule, the dispersion is influenced by a great number of causes, and the influence of each of these causes is small in comparison with the total influence of all others. In these conditions the law of distribution should be close to the normal. For the distribution of deviations obtained as a result of mathematical firing, the hypothesis about normal character of distribution can be subjected to a check by using either Pearson criterion χ^2 of the criterion of Kolmogorov. The method of application of these criteria is not described here.

The method of mathematical firing has certain deficiencies along with a number of merits. The basic one is the impossibility to separate the influence on the dispersion of separate perturbing factors, since during the calculation of perturbed trajectories there appears only their joint action. Therefore it is difficult to determine with which of the causes effecting dispersion one should struggle first if this dispersion is excessively great. In connection with this the method of mathematical firings more frequently is used as a checking method, and design calculations of dispersion are usually conducted by other methods similar to those above described. However, even in certain design calculations the method of mathematical firings can be useful. Let us assume, for example, that it is required to compare several laws of range control and to select from them the optimum, just as in § 37 the optimum coefficient of compensation of the integrator was selected. For this purpose it is possible to conduct several series of mathematical firings with their law of range control in each series. For each series there is determined the standard deviation in range σL, and that law is selected with which σL is found to be minimum.

If in each of such series the values of the perturbing factors are assigned independently of the other series, then the value σL, besides the law of control, will be influenced by the set of values of perturbing factors accepted for the given series. As a result the selection of the optimum law can appear erroneous. In order to considerably decrease the probability of such an error, it is expedient to use the same set of values of perturbing factors in all series. More precisely, for different trajectories within the first series the perturbing factors are selected randomly and independently of each other. The selected values of these factors are memorized. With calculation of the i-th trajectory (i = 1, 2, ..., N) of any subsequent series, values of perturbing factors will be taken the same as for the i-th trajectory of the first series. As a result the dependence of dispersion on the form of the law of control becomes more clear. If these comparable laws of control are distinguished only the numerical value of one or several parameters (for example, the coefficient of compensation of the integrator, the angle of setting of its sensing device, etc.), then the dependence of characteristics of dispersion on these parameters appears smooth and optimum values of the parameters are easily found.

In many cases for an appraisal of the dispersion and in the solution of other problems connected with the influence of small deviations of design parameters on the range of flight, there can appear useful approximate formulas allowing the calculation of range derivatives by design parameters not resorting to numerical integration. Such formulas possessing an acceptable accuracy can be obtained on the basis of the method of approximation of calculation of the flying range expounded in § 24. Let us find at first partial derivatives of range according to quantities N and N_1. Differentiating formulas (24.15) expressing x_C and y_C by N, N_1, φ_0 and t_C, we will obtain:

$$\begin{aligned} dx_C &= g\cos\varphi_0(t_C\,dN - dN_1) + \\ &\quad + gN\cos\varphi_0\,dt_C - g(Nt_C - N_1)\sin\varphi_0\,d\varphi_0, \\ dy_C &= g\sin\varphi_0(t_C\,dN - dN_1) + \\ &\quad + g(N\sin\varphi_0 - t_C)\,dt_C + g(Nt_C - N_1)\cos\varphi_0\,d\varphi_0. \end{aligned}$$

Let us exclude hence $d\varphi_0$:

$$\cos\varphi_0\,dx_C + \sin\varphi_0\,dy_C = g(t_C\,dN - dN_1) + g(N - t_C\sin\varphi_0)\,dt_C.$$

The coefficient at dt_C, on the basis of formula (24.16), turns into zero. Differentiating the relation (24.24), we find

$$dy_C = -\frac{x_C}{R}\,dx_C.$$

We substitute this expression for dy_C in the preceding equation

$$\left(\cos\varphi_0 - \frac{x_C}{R}\sin\varphi_0\right)dx_C = g(t_C\,dN - dN_1)$$

or, on the basis of formulas (24.17) and (24.22),

$$dx_C = \frac{g\left(\frac{N}{\sin\varphi_0}\,dN - dN_1\right)}{\cos\varphi_0 - b\cos\varphi_0(1 - a\sin\varphi_0)}.$$

Consequently,

$$\frac{dx_C}{dN} = \frac{gN}{\sin\varphi_0\cos\varphi_0(1 - b + ab\sin\varphi_0)}. \qquad (39.23)$$

$$\frac{\partial x_C}{\partial N_1} = -\frac{g}{\cos\varphi_0\,(1 - b + ab\sin\varphi_0)}. \tag{39.24}$$

The derivative of coordinate x_C by the design parameter λ_k can be calculated by the formula

$$\frac{\partial x_C}{\partial \lambda_k} = \frac{\partial x_C}{\partial N}\,\frac{\partial N}{\partial \lambda_k} + \frac{\partial x_C}{\partial N_1}\,\frac{\partial N_1}{\partial \lambda_k},$$

where partial derivatives $\partial N/\partial\lambda_k$ and $\partial N_1/\partial\lambda_k$ can be determined by differentiation of formulas (24.30) and (24.31) according to design parameters entering into them:

$$\begin{aligned}
\frac{\partial N}{\partial P_{\text{уд}\,i}} &= \ln\frac{T_i - t_{\text{к}\,i-1}}{T_i - t_{\text{к}\,i}},\\
\frac{\partial N}{\partial T_i} &= P_{\text{уд}\,i}\left(\frac{1}{T_i - t_{\text{к}\,i-1}} - \frac{1}{T_i - t_{\text{к}\,i}}\right) =\\
&\qquad\qquad = -\frac{P_{\text{уд}\,i}\,(t_{\text{к}\,i} - t_{\text{к}\,i-1})}{(T_i - t_{\text{к}\,i-1})(T_i - t_{\text{к}\,i})},\\
\frac{\partial N}{\partial t_{\text{к}\,i}} &= \frac{P_{\text{уд}\,i}}{T_i - t_{\text{к}\,i}} - \frac{P_{\text{уд}\,i+1}}{T_{i+1} - t_{\text{к}\,i}},\\
\frac{\partial N_1}{\partial P_{\text{уд}\,i}} &= T_i \ln\frac{T_i - t_{\text{к}\,i-1}}{T_i - t_{\text{к}\,i}} + t_{\text{к}\,i-1} - t_{\text{к}\,i},\\
\frac{\partial N_1}{\partial T_i} &= P_{\text{уд}\,i}\left[\ln\frac{T_i - t_{\text{к}\,i-1}}{T_i - t_{\text{к}\,i}} - \frac{T_i\,(t_{\text{к}\,i} - t_{\text{к}\,i-1})}{(T_i - t_{\text{к}\,i-1})(T_i - t_{\text{к}\,i})}\right],\\
\frac{\partial N_1}{\partial t_{\text{к}\,i}} &= \left(\frac{P_{\text{уд}\,i}}{T_i - t_{\text{к}\,i}} - \frac{P_{\text{уд}\,i+1}}{T_{i+1} - t_{\text{к}\,i}}\right) t_{\text{к}\,i}.
\end{aligned}$$

In these formulas one should consider

$$\begin{gathered}
t_{\text{к}\,0} = 0,\\
P_{\text{уд}\,m+1} = 0.
\end{gathered}$$

If range is determined by the formula (24.28), then for transition from $\partial x_C/\partial\lambda_k$ to $\partial L/\partial\lambda_k$ one should use formula

$$\frac{\partial L}{\partial \lambda_k} = \frac{dL}{dx_C}\,\frac{\partial x_C}{\partial \lambda_k} = \frac{1}{1 + \left(\frac{x_C}{2R}\right)^2}\,\frac{\partial x_C}{\partial \lambda_k}.$$

§ 40. Maximum Range of Firing

To the number of problems connected with the influence on the flight of the rocket of small deviations of different factors belongs the problem of the determination of the maximum range of firing.

Let us consider a single-stage rocket whose propulsion system uses fuel consisting of two components, an oxidizer and fuel. Each of these components is placed in its own tank.

For achievement of an assigned flying range L such a rocket should expend a definite quantity of oxidizer and fuel. However when one considers a whole series of rockets of identical construction and with identical nominal characteristics, then it will appear that each of them uses a different quantity of both oxidizer and fuel for achievement of the same flying range. This is explained by the fact that the values of basic technical characteristics of the rocket and also conditions of its flight are subject to random scattering.

To the number of characteristics of the rocket essentially affecting quantities of the components of fuel necessary for achievement of the assigned flying range belong the weight of construction of the rocket, weights of the oxidizer and fuel filled in tanks of the rocket before its launching, specific thrust of the engine, total flow rate per second of fuel (oxidizer and fuel together), relations of flow rates of components of fuel, and others. Of the external factors the role of temperature and density of air, wind etc. can be factors.

Random scattering of all the values named leads to the fact that remainders of components of fuel in the tanks at the moment when the engine of the rocket is turned off are also subject to scattering, i.e., are random variables. It follows from this that the computed value of remainders of the oxidizer and fuel for nominal characteristics of the rocket and nominal conditions of its flight should not be too small. Otherwise the rocket cannot reach the assigned range because of premature expenditure of one of the components of fuel.

To each value of the sighting range of flight L there can be set into conformity the probability P(L) of the fact that reserves of components of fuel will appear sufficient for achievement of this distance. The less the probability P(L) the greater the sighting range of firing L, i.e., the distance at which the instrument sending command for turning off the engine is tuned. If one were to assign certain probability P_0 quite close to unity, then the value of range $L_{\text{пр}}$ for which $P(L_{\text{пр}}) = P_0$ is called the maximum range of firing corresponding to the reliability P_0. Let us emphasize that the discussion concerns the distance accessible almost by any rocket of the examined series. The rocket separately taken, with a favorable combination of design parameters and in favorable conditions, can fly at a distance considerably greater than $L_{\text{пр}}$. On the other hand, a certain insignificant portion of the rockets $(1 - P_0)$ cannot reach this distance. The maximum range $L_{\text{пр}}$ characterizes the whole series or a given type of rocket as a whole and does not have a direct relation to the maximum possible flying range of the separate rocket.

Nominal remainders of components of fuel in the tanks (i.e., remainders calculated for nominal values of characteristics of the rocket and nominal external conditions of the flight) corresponding to maximum ranges of firing are called guaranteed reserves of components of fuel. In other words, guaranteed reserves of fuel are such reserves whose presence in tanks of the rocket during its motion about the nominal trajectory provides achievement of the assigned flying range with the probability P_0.

The problem of the determination of the maximum range of firing and guaranteed reserves of fuel can be solved by different methods. A more accurate method is based on the obtaining of the dependence P(L) and solution of equation $P(L) = P_0$. We will not proceed this way, since it leads to bulky analytical calculations and time-consuming numerical calculations. Let us dwell on another less accurate method which is simple in its calculation scheme.

We will consider that factors λ_i affecting the trajectory of the flight of the rocket experience only small deviations $\Delta\lambda_i$ leading to small deviations in the flying range δL and remainders of the oxidizer $\delta G_{\text{ок}}^{(\text{ост})}$ and fuel $\delta G_{\text{г}}^{(\text{ост})}$ in tanks, which the connection between these small deviations is quite accurately described by linear equations of the form

$$\delta L = l_1\Delta\lambda_1 + l_2\Delta\lambda_2 + \ldots + l_n\Delta\lambda_n. \tag{40.1}$$

$$\delta G_{\text{ок}}^{(\text{ост})} = a_1\Delta\lambda_1 + a_2\Delta\lambda_2 + \ldots + a_n\Delta\lambda_n. \tag{40.2}$$

$$\delta G_{\text{г}}^{(\text{ост})} = b_1\Delta\lambda_1 + b_2\Delta\lambda_2 + \ldots + b_n\Delta\lambda_n. \tag{40.3}$$

In these formulas it is assumed that deviations δL, $\delta G_{\text{ок}}^{(\text{ост})}$ and $\delta G_{\text{г}}^{(\text{ост})}$ correspond to the constant time of operation of the engine, which is equal to the nominal time of work t_1 necessary for achievement of the assigned distance during flight on on the undisturbed trajectory. For achievement of the same distance during motion along the perturbed trajectory it is necessary to change the time of operation of the engine. Derivatives of values L, $G_{OK}^{(OCT)}$, $G_{\Gamma}^{(OCT)}$ in time of operation of the engine will be designated by a dot. Deviations of these values from the nominal values, taking into account the change in time of operation of the engine will be designated ΔL, $\Delta G_{OK}^{(OCT)}$ and $\Delta G_{\Gamma}^{(OCT)}$. For them these expressions are correct

$$\begin{gathered}
\Delta L = \delta L + \dot{L}\Delta t_1, \\
\Delta G_{\text{ок}}^{(\text{ост})} = \delta G_{\text{ок}}^{(\text{ост})} + \dot{G}_{\text{ок}}^{(\text{ост})}\Delta t_1 = \delta G_{\text{ок}}^{(\text{ост})} - \dot{G}_{\text{ок}}\Delta t_1. \\
\Delta G_{\text{г}}^{(\text{ост})} = \delta G_{\text{г}}^{(\text{ост})} + \dot{G}_{\text{г}}^{(\text{ост})}\Delta t_1 = \delta G_{\text{г}}^{(\text{ост})} - \dot{G}_{\text{г}}\Delta t_1.
\end{gathered}$$

where $\dot{G}_{OK}$ and $\dot{G}_{\Gamma}$ designate the flow rates per second of the oxidizer and fuel:

$$\begin{gathered}
\dot{G}_{\text{ок}} = |\dot{G}_{\text{ок}}^{(\text{ост})}|. \\
\dot{G}_{\text{г}} = |\dot{G}_{\text{г}}^{(\text{ост})}|.
\end{gathered}$$

The change in time of operation of the engine on the perturbed trajectory Δt_1 will be determined from the condition of constancy of the flying range

$$\Delta L = 0,$$

whence

$$\Delta t_1 = -\frac{\delta L}{\dot{L}}.$$

Consequently,

$$\begin{gathered}
\Delta G_{\text{ок}}^{(\text{ост})} = \delta G_{\text{ок}}^{(\text{ост})} + \frac{\dot{G}_{\text{ок}}}{\dot{L}}\delta L. \\
\Delta G_{\text{г}}^{(\text{ост})} = \delta G_{\text{г}}^{(\text{ост})} + \frac{\dot{G}_{\text{г}}}{\dot{L}}\delta L.
\end{gathered}$$

Using formulas (40.1)-(40.3), we obtain

$$\Delta G_{\text{ок}}^{(\text{ост})} = \sum_{k=1}^{n}\left(a_k + \frac{\dot{G}_{\text{ок}}}{\dot{L}}l_k\right)\Delta\lambda_k. \tag{40.4}$$

$$\Delta G_{\text{г}}^{(\text{ост})} = \sum_{k=1}^{n}\left(b_k + \frac{\dot{G}_{\text{г}}}{\dot{L}}l_k\right)\Delta\lambda_k. \tag{40.5}$$

In practically all the encountered cases we can assume that deviations $\Delta\lambda_k$ of factors affecting the flying range and remainders of components of fuel in the tanks are independent random quantities subordinated to the normal law of distribution with the mean value zero and with the standard deviation $\sigma\lambda_k$. With this assumption from formulas (40.4) and (40.5) there ensue the following expressions for standard deviations of remainders of the oxidizer and fuel:

$$\sigma G_{\text{ок}}^{(\text{ост})} = \sqrt{\sum_{k=1}^{n}\left(a_k + \frac{\dot{G}_{\text{ок}}}{\dot{L}}l_k\right)^2(\sigma\lambda_k)^2}. \tag{40.6}$$

$$\sigma G_{\text{г}}^{(\text{ост})} = \sqrt{\sum_{k=1}^{n}\left(b_k + \frac{\dot{G}_{\text{г}}}{\dot{L}}l_k\right)^2(\sigma\lambda_k)^2}. \tag{40.7}$$

From formulas (40.4) and (40.5) it is possible also to conclude that remainders of components of fuel are random variables having a normal law of distribution, and that their mean values are equal to zero (see § 31).

Let us assign a certain probability P_0 close to unity. From the equation

$$P_0 = \int_{-k\sigma}^{k\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}} dz = \int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

one can determine such value k that with the probability P_0 the random deviation of quantity ξ, obeying the normal law of distribution with the mean value zero and standard deviation σ, in absolute value will not exceed $k\sigma$. As was already mentioned in § 31, the most commonly used are values $P_0 \approx 0.997$ corresponding to $k = 3$ and $P_0 \approx 0.993$, to which the value $k \approx 2.698 \approx 2.7$ corresponds.

As guaranteed reserves of the oxidizer and combustible are the values

$$G_{\text{ок}}^{(\text{гар})} = k\sigma G_{\text{ок}}^{(\text{ост})} \tag{40.8}$$

and

$$G_{\text{г}}^{(\text{гар})} = k\sigma G_{\text{г}}^{(\text{ост})} \tag{40.9}$$

respectively, where $\sigma G_{OK}^{(OCT)}$ and $\sigma G_{\Gamma}^{(OCT)}$ are standard deviations of remainders of components determined by the formulas (40.6) and (40.7). Let us estimate the probability of achievement of the assigned distance L during flight about any trajectory, if reserves of components of fuel are selected so that during flight for this distance along the nominal trajectory remainders of components at the time of the turning off of the engine have values determined by the formulas (40.8) and (40.9).

Distance L can be attained with flight on a certain perturbed trajectory, if remainders of the oxidizer $G_{OK}^{(OCT)}$ and fuel $G_{\Gamma}^{(OCT)}$, calculated for the moment of the turning off of the engine on this trajectory corresponding to the flying range L, are found to be positive. Let us consider each component of fuel separately. With the probability P_0 the random deviation of the remainder of the component will not exceed $k\sigma G^{(OCT)}$. In this case the remainder of the component is knowingly positive, since the nominal value of the remainder is accepted equal to $k\sigma G^{(OCT)}$. If, however, the deviation of the remainder exceeds $k\sigma G^{(OCT)}$, then with respect to a shortage of this component only the case of a negative deviation is dangerous. In virtue of the symmetry of the normal law of distribution the probability of shortage of a given component of fuel is equal to $\frac{1}{2}(1 - P_0)$.

There are four possible results of the launching of a rocket:

1) $G_{\text{ок}}^{(\text{ост})} > 0, \quad G_{\text{г}}^{(\text{ост})} > 0;$
2) $G_{\text{ок}}^{(\text{ост})} > 0, \quad G_{\text{г}}^{(\text{ост})} < 0;$
3) $G_{\text{ок}}^{(\text{ост})} < 0, \quad G_{\text{г}}^{(\text{ост})} > 0;$
4) $G_{\text{ок}}^{(\text{ост})} < 0, \quad G_{\text{г}}^{(\text{ост})} < 0.$

Their probabilities will be designated respectively by P_1, P_2, P_3 and P_4. The sum of these probabilities should be equal to one

$$P_1 + P_2 + P_3 + P_4 = 1. \tag{40.10}$$

The shortage of an oxidizer, having according to the proven the probability $\frac{1}{2}(1 - P_0)$, is encountered at the third and fourth results, so that

$$P_3 + P_4 = \frac{1}{2}(1 - P_0). \tag{40.11}$$

Analogously the probability of the shortage of fuel is equal to

$$P_2 + P_4 = \frac{1}{2}(1 - P_0). \tag{40.12}$$

Subtracting from equality (40.10) the sum of equalities (40.11) and (40.12) we obtain

$$P_1 - P_4 = 1 - (1 - P_0),$$

whence

$$P_1 = P_0 + P_4 > P_0.$$

But P_1 is the probability of only a favorable result, i.e., the probability of achievement of the assigned flying range. Thus the expounded method of determination of guaranteed reserves of fuel in the case of a single-stage rocket, for which the reserve of fuel is placed in two tanks, provides achievement of the assigned flying range with a probability though not exactly equal to P_0 but in any case not smaller than P_0. Because of its simplicity this method is used in more complicated cases, i.e., for rockets with more than two tanks (the number of tanks and not the number of different components of fuel is important, since the guaranteed reserve should be foreseen in each separate tank). But in these cases it no longer allowed to affirm that the probability of achievement of maximum range will not be lower than P_0.

In conclusion of this paragraph let us touch upon the method of determination of coefficients a_k and b_k in formulas (40.2) and (40.3). Coefficients l_k, characterizing the influence of different perturbing factors on the flying range, coincide with coefficients z_{4k} introduced in § 36 and are calculated by methods of the preceding paragraphs. Calculation of coefficients a_k and b_k is usually made considerably simpler. Thus, for example, for the remainder of the oxidizer at the time t_1 it is possible to write the expression

$$G_{\text{ок}}^{(\text{ост})} = G_{\text{ок}}^{(0)} - t_1 \dot{G}_{\text{ок}},$$

where $G_{OK}^{(0)}$ is the weight of the oxidizer filled in the tank of the rocket before the launch. It is expedient to express the flow rate of the oxidizer $\dot{G}_{OK}$ in terms of total fuel consumption $\dot{G}$ and the relation of flow rates of components

$$k = \frac{\dot{G}_{\text{ок}}}{\dot{G}_{\text{г}}}.$$

The fact is that for liquid-propellant rocket engines the quantities $\dot{G}$ and k can be examined as independent random quantities, while the flow rates of components $\dot{G}_{OK}$ and $\dot{G}_{\Gamma}$ are connected by a rather substantial correlation dependence. The expression for $\dot{G}_{OK}$ in terms of $\dot{G}$ and k, obviously, has the form

$$\dot{G}_{\text{ок}} = \frac{k}{k+1}\dot{G},$$

so that

$$G_{\text{ок}}^{(\text{ост})} = G_{\text{ок}}^{(0)} - \frac{k}{k+1}\dot{G}t_1 \tag{40.13}$$

and, analogously,

$$G_{г}^{(ост)} = G_{г}^{(0)} - \frac{1}{k+1}\dot{G}t_1. \tag{40.14}$$

Differentiating these dependences and replacing the differentials by finite increments, we will obtain for t_1 = const:

$$\delta G_{ок}^{(ост)} = \Delta G_{ок}^{(0)} - \frac{kt_1}{k+1}\Delta\dot{G} - \frac{\dot{G}t_1}{(k+1)^2}\Delta k. \tag{40.15}$$

$$\delta G_{г}^{(ост)} = \Delta G_{г}^{(0)} - \frac{t_1}{k+1}\Delta\dot{G} + \frac{\dot{G}t_1}{(k+1)^2}\Delta k. \tag{40.16}$$

Quantities $G_{ок}^{(0)}$, $G_{г}^{(0)}$, $\dot{G}$ and k should be included in the number of parameters λ_k. The formulas obtained are a concrete recording of relations which in general form were represented by formulas (40.2) and (40.3). If any of parameters λ_k, for example, specific thrust $P_{уд}$, does not directly affect remainders of components of fuel (at fixed t = t_1), then the corresponding coefficients a_k and $b_к$ in formulas (40.2) and (40.3) should be considered equal to zero. In exactly the same way one should consider coefficient l_k equal to zero with a deviation of such parameter λ_k, which enters into formulas (40.2) and (40.3) but does not affect the flying range with a constant time of operation of the engine (for example, the relation of flow rates of k components).

For different concrete schemes of engine installations dependences (40.2) and (40.3) can appear more complicated and contain a greater quantity of different factors than in formulas (40.15) and (40.16), but this fundamentally changes nothing in the method of calculation of the guaranteed reserves of fuel.

PART FOUR

SELECTING THE FORM OF THE TRAJECTORY

CHAPTER X

FORMULATION OF THE PROBLEM OF THE SELECTION OF PROGRAM

§ 41. Requirements for the Program

In this part basic questions connected with the selection of the program of the pitch angle will be discussed.

The program of the pitch angle, and sometimes and simply the program, is the law of the change in the angle of inclination of the axis of the rocket. Sometimes the program is called the law of change in the angle produced by the programer. Angles of inclination of the axis of the rocket and angles assigned by the programer do not coincide. However, the influence of this noncoincidence on the basic properties of the trajectory is insignificant. Therefore, in all ballistic calculations except calculations by the most general equations of motion (§§ 14, 16), it is assumed that the axis of the rocket strictly fulfills angle turns assigned to it by the programer.

Usually the law of change in the pitch angle is set depending upon the time. The law of change produced by the programer is designated in the form of the function $\varphi_{\text{пр}}(t)$, and the true change of the angle of inclination of the axis is recorded in the form of $\varphi(t)$. As was already mentioned, we will consider that $\varphi(t) = \varphi_{\text{пр}}(t)$.

Equations of motion contain a program as an assigned function. Therefore, final results of integration of equations of motion, i.e., $v_{\text{к}}$, $y_{\text{к}}$, $x_{\text{к}}$, $\theta_{\text{к}}$, the full distance L or other characteristics interesting to us to a considerable degree are determined by function $\varphi(t)$. The selection of this function is directly influenced by three basic factors: design parameters of the rocket, peculiarities of control system, and problems posed before the trajectory with the launching of the rocket.

If, for example, it is required to select trajectories providing ultimate range for two different rockets with an identical control systems, then this will require application of different programs of $\varphi(t)$. For the formation of trajectories providing minimum dispersion for the same rocket different programs are required, if one were to proceed from different principles of the range control of firing. The same rocket with the assigned control system requires application of different programs of the pitch angle depending upon whether it is required to provide maximum range, minimum dispersion, maximum altitude of flight or some other quality of the trajectory.

Thus to give a single rule for the selection of a program useful for all possible cases is impossible. However there are certain general principles which one should follow almost in all cases. We will dwell on them more concretely.

In Chapter V dependences were derived allowing the judging of the influency of the finite angle $\theta_{\text{к}}$ on range. In the same chapter it was shown that for every pair of values $v_{к}$ and $h_{к}$ there can be found angle $\theta_{к}$ at which the distance will be maximum. Such an angle was called optimum because it permits the best use of the energy acquired by the rocket on the powered section. But the values themselves $v_{к}$ and $h_{к}$ depend on the angle $\theta_{к}$ and function $\varphi(t)$.

In most cases, proceeding from requirements for the trajectory, the function $\varphi(t)$ should be selected in such a way that the range is obtained the greatest possible. However, this does not mean that before the program is placed the problem of the achievement of maximum theoretical range as obligatory and essential, although flying range is one of the most important tactical characteristics. The requirement of obtaining maximum range when necessary should be subordinated to other more important requirements whose fulfillment is technically more complicated than the achievement of the assigned range. To them, in the first place, one should relate the requirement of minimum dispersion. Therefore, the problem of the selection of a program is the determination of such function $\varphi(t)$, which for the assigned rocket with the accepted method of control, in particular, the method of turning off the engine, would provide the assigned distance insignificantly differing from the maximum with minimum dispersion.

At present several solutions of the variational problem are known according to the selection of optimum (from the point of view of obtaining range) program, but the majority of them is obtained with certain simplifying assumptions or for particular cases of motion and neglecting peculiarities of the defined control system and method of turning off the engine.

But even in the presence of a common solution of the problem in our setting it was necessary to check the obtained program from the point of view of fulfillment of a number of requirements imposed by conditions of strength, stability, convenience of exploitation and others. Consequently, for the solution of the variational problem it is necessary to impose corresponding additional limitations from conditions of fulfillment of the mentioned requirements.

To such limitations, in the first place, pertain the following:

1) vertical launch and definite duration of vertical flight;

2) continuity $\varphi(t)$, $\dot{\varphi}(t)$, $\ddot{\varphi}(t)$ and limitedness $\ddot{\varphi}(t)$;

3) limitedness of normal G-forces;

4) zero angles of attack at speeds close to sound;

5) special conditions caused by the method of control and turning off of the engine;

6) firing at any distance in an assigned range with one or a minimum number of programs.

Let us turn to the consideration of causes which are caused by requirements mentioned.

1. Vertical launch is the most convenient and simple and requires no special directrixes and other devices and apparatuses. To set the rocket vertically is considerably easier than to set it exactly at an assigned angle.

Besides this, in a vertical launch there is a minimum of lateral shifts of the rocket, which can take place with a slanted launch in the first seconds of flight.

Duration of the vertical section is determined mainly by the time necessary for the controls to be sufficiently effective. This in turn is determined by engine performance.

2. The requirement on continuity $\varphi(t)$, $\phi(t)$, $\ddot{\varphi}(t)$ and limitedness $\ddot{\varphi}(t)$ is conditioned by possibilities of instruments and controls. Really, a break in function

$\varphi(t)$ contradicts the physical meaning of the program, and a break in function $\dot{\varphi}(t)$ (or break in curve $\varphi(t)$) corresponds to infinite controlling moments. A break in functions $\ddot{\varphi}(t)$ corresponds to an instantaneous change in moments, i.e., angles of deviation of control surfaces or infinite angular velocities of control surfaces. The limitedness $\ddot{\varphi}(t)$ is dictated by limited possibilities of controls, since the maximum value of $\ddot{\varphi}(t)$ is determined by the maximum deviation of the control surfaces.

Thus the program assigned to a certain control circuit requires fulfillment of conditions of § 2. In certain cases the exact observance of these requirements can be refused if the appearing mismatches between φ and $\varphi_{пр}$ will not have considerable influence on further flight, since they will be able to be depleted by the control system for a sufficiently short interval of time.

3. Axial loads on the rocket are determined mainly by parameters, namely v_0 and μ_K. Therefore, the program cannot render considerable influence on G-forces in an axial direction. Regarding transverse G-forces, they depend mainly on the magnitude of the aerodynamic moment, which is closely connected with angles of attack and, consequently, with the program.

This circumstance imposes on the program the requirement limiting the magnitude of the aerodynamic moment determined by the product

$$M_{aa_1} = c_y' qS(\varphi - \theta)(x_a - x_т)$$

(formulas 11.17) and (11.19)).

Calculations show that substantial change in the moment can be reached only owing to angles α, since the change $c_y' = (\partial c_y/\partial\alpha)$ and $q = (\rho v^2/2)$ on the trajectory depends on the program in a much lesser degree. Thus with calculation of the program it is necessary to limit angles of attack in such a manner that obtained aerodynamic moments do not require too durable and heavy construction. It is clear that this requirement with respect to the magnitude of allowed angles of attack pertains mainly to sections of the trajectory with high velocity heads. It is desirable to pass these sections with miminum or zero angles of attack.

4. As a rule, effectiveness of controls does not depend on the speed of the rocket and conditions of the flowing around. But the region of speeds (Mach numbers) $M = 0.8$-1.2 is characterized by a sharp change in aerodynamic coefficients. For the operation of controls coefficients $\partial c_y/\partial\alpha$ and $\partial m_z/\partial\alpha$ have singular value. Desiring to reduce the influence of sharp changes of these coefficients to a minimum, it is necessary to take care that the indicated region be passed with zero angles of attack.

5. Requirements of this point are not general and in an identical measure are obligatory for all rockets. Depending upon conditions of the operation of systems of measurements and instruments of control of the rocket and also the providing of definite properties of the trajectory there can appear special requirements for the program, for example, the requirement of providing rectilinearity of the trajectory on some segment, limitation in assigned limits of the angle between the axis of the rocket and communication line of the rocket with the ground center, motion at constant angle pitch, and a number of similar requirements.

6. This point provides the possibility of firing at all distances in the assigned range with one or a minimum number of programs. For rockets possessing comparatively small distances (up to 1500-2000 km) or greater distances but in a quite narrow range, this condition is satisfied comparatively easy, since optimum programs connot differ greatly from each other.

For rockets, possessing a wide range of distances, when it is impossible to select a quite satisfactory program (one) for all distances, it can be necessary to divide the range into several smaller ranges. In this case it is necessary to try to bring the number of ranges to a minimum.

The requirement of this point, just as that of the preceding, is not obligatory for all rockets.

§ 42. Maximum Range and Minimum Dispersion

Let us consider in the common form the conditions of obtaining the maximum range and minimum dispersion.

In the solution of the problem of the achievement of maximum range we should proceed from the fact that the rocket possesses a definite reserve of fuel, which is completely expended during acceleration on the powered-flight trajectory. This quantity of fuel at nominal values and design parameters of the rocket can be set in conformity to the definite time of operation of the engine $t_{к}$. Let us consider for simplicity the plane motion (analogous reasoning can be conducted for spatial motion) with which the flying range can be expressed as a function of four kinematic parameters, for example, speed, angle of its inclination to the horizon and of the two coordinates taken at the time of the turning off of the engine:

$$L = f(v_{к}, \theta_{к}, x_{к}, y_{к}). \tag{42.1}$$

We will modify the program of the angle of pitch, keeping all other parameters of the rocket constant. This means that instead of the motion of the rocket on the nominal trajectory with $\varphi = \varphi_{пр}(t)$ there is examined the motion with $\varphi = \varphi_{пр}(t) + \delta\varphi(t)$, where $\delta\varphi(t)$ is the arbitrary deviation (error in the fulfillment of the program), possible in real conditions. With this we obtain the variation of parameters of motion at the end of the powered-flight section and, consequently, the variation of the full range. In conformity with that said above, these variations should be taken for the fixed moment $t_{к}$ corresponding to the complete expenditure of fuel, and the necessary condition of achievement of maximum range can be recorded in the form

$$\delta L = \frac{\partial L}{\partial v}\delta v\Big|_{t=t_{к}} + \frac{\partial L}{\partial \theta}\delta\theta\Big|_{t=t_{к}} + \frac{\partial L}{\partial x}\delta x\Big|_{t=t_{к}} + \frac{\partial L}{\partial y}\delta y\Big|_{t=t_{к}} = 0. \tag{42.2}$$

Before writing down the condition of minimum dispersion, let us note that in actual flight with an operating range control system the deviation of the impact point from the assigned does not depend directly on the maximum time of operation of the engine $t_{к}$, but is determined by deviations of parameters of motion $v_{к}$, $\theta_{к}$, $x_{к}$, and $y_{к}$ at the time of the actual turning off of the engine on command from automatic range control device. Thus, designating these deviations $\Delta v_{к}$, $\Delta\theta_{к}$, $\Delta x_{к}$, and $\Delta y_{к}$, the condition of minimum dispersion will be written thus:

$$\Delta L = \frac{\partial L}{\partial v}\Delta v_{к} + \frac{\partial L}{\partial \theta}\Delta\theta_{к} + \frac{\partial L}{\partial x}\Delta x_{к} + \frac{\partial L}{\partial y}\Delta y_{к} = 0. \tag{42.3}$$

More precisely, this is the condition of minimum influence of deviations of the pitch program $\delta\varphi(t)$ on the deviation of the point of impact with an operating range control system.

In general the turnings off of the engine by one of the possible methods (with the help of the integrator of axial G-forces, with the achievement of the assigned speed, with the achievement of the assigned combination of coordinates and speed, etc.) of the variation of parameters of motion at the end of the powered-flight section will be composed of variations of parameters at the calculated moment of turning off of the engine $t_{к}$ and variations induced by a change in time of the turning off of the engine $\Delta t_{к}$ so that

$$\left.\begin{aligned}\Delta v_{\text{к}} &= \delta v\,\big|_{t=t_{\text{к}}} + \frac{\partial v}{\partial t}\Delta t_{\text{к}},\\ \Delta\theta_{\text{к}} &= \delta\theta\,\big|_{t=t_{\text{к}}} + \frac{\partial \theta}{\partial t}\Delta t_{\text{к}},\\ \Delta x_{\text{к}} &= \delta x\,\big|_{t=t_{\text{к}}} + \frac{\partial x}{\partial t}\Delta t_{\text{к}},\\ \Delta y_{\text{к}} &= \delta y\,\big|_{t=t_{\text{к}}} + \frac{\partial y}{\partial t}\Delta t_{\text{к}}\end{aligned}\right| \tag{42.4}$$

(let us recall that both these and other variations are caused in the examined case only by a variation of the pitch angle $\delta\varphi(t)$).

Substituting these variations into expression (42.3), the condition of minimum dispersion will be obtained in the following form:

$$\Delta L = \frac{\partial L}{\partial v}\delta v\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial \theta}\delta\theta\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial x}\delta x\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial y}\delta y\Big|_{t=t_{\text{к}}} + \\ + \left(\frac{\partial L}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial L}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial L}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial t}\right)\Delta t_{\text{к}} = 0.$$

or

$$\Delta L = \frac{\partial L}{\partial v}\delta v\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial \theta}\delta\theta\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial x}\delta x\Big|_{t=t_{\text{к}}} + \\ + \frac{\partial L}{\partial y}\delta y\Big|_{t=t_{\text{к}}} + \frac{\partial L}{\partial t}\Delta t_{\text{к}} = 0. \tag{42.5}$$

Comparing conditions of maximum range (42.2) and minimum dispersion (42.5), we arrive at the conclusion that in general these conditions are not identical, and they cannot be feasible simultaneously.

If the condition of the maximum range does not depend on the method of the turning off, then the condition of minimum dispersion depends on the method of turning off of the engine, since Δt_K will be determined namely by the method of the turning off of the engine. Only in one particular case, namely, when turning off of the engine is produced after achievement of the assigned time of operation, do these conditions completely coincide and, consequently, are fulfilled simultaneously.

This by no means mean that such a method of turning off is good, and only signifies the fact that of all the possible programs selected for such a method of turning off the best in the sense of accuracy will be that one which simultaneously corresponds to the maximum range. The very method of turning off by time is not applied in practice in view of the extremely great methodical errors peculiar to it.

Since with all other possible methods of the turning off of the engine conditions of maximum range and minimum dispersion do not coincide, it is necessary during calculation of the concrete program of the pitch angle to assign a condition whose fulfillment should be provided in the first place, and fulfillment of the second condition can be only checked; more correctly not the second condition but the degree of deviation from it can be checked. In practice most frequently it is necessary to discover a certain compromise solution, giving satisfactory accuracy and at the same time not very great loss in range comparatively with the highest possible.

So that peculiarities of the selection of a program become clearer, it is necessary to dwell on one more question. The fact is that it is important to provide fulfillment of the condition of minimum dispersion not only for the upper limit of the assigned range of distances of firing but also for any distance starting with the minimum. In the opposite case firing at lesser distances will be produced with greater errors than that for greater distances.

In principle such a problem is feasible the more so because the condition of providing maximum range drops with the selection of programs for firing at any distances except the region of maximum ranges. Thus the program providing fulfillment of the condition of minimum dispersion on the whole range of distances (and without great losses in the maximum range) would be best.

The solution of the corresponding variational problem can determine the program satisfying the selected conditions only for one distance. If for this distance we take the maximum, then for any other distance lying between the maximum and minimum, the obtained program will not provide minimum dispersion, since for every distance solution of the variational problem will give its program different from that of others. Thus we arrive at the conclusion that the solution of the variational problem in principle does not permit selecting such a program which would give minimum dispersion on the all range of distances. It is possible only by finding one of solutions to check it for other distances for the purpose of clarification of limits of applicability of one program.

Irrespective of what the method of turning off of the engine and what the program providing minimum dispersion, the maximum range in all cases is checked by the calculation.

CHAPTER XI

METHODS OF SELECTION OF A PROGRAM

§ 43. Selection of a Program of Maximum Range

Let us discuss the applied procedures of determination of the maximum range of the rocket. Let us note that the exact solution of the problem on maximum range is not obtained in final form. However, there is known a number of solutions obtained with certain simplifying assumptions, which give good orientation for the selection of a program of maximum range in real conditions of the motion.

In § 24 there is examined the variational problem by definition of the program of maximum range under conditions of a plane-parallel field of forces and the absence of atmosphere. It is shown that a certain constant direction of traction of the engine, depending on the basic design parameters of the rocket, realizes a maximum of distance.

Examined in article [11] is the variational problem by selection of the program of pitch angle providing maximum horizontal speed at an assigned altitude.

The problem is solved on the assumption that motion occurs outside the outside the atmosphere in a plane-parallel field of forces. As a result of the solution it is obtained that the tangent of the pitch angle with the optimum program should be a linear function of time, i.e.,

$$\operatorname{tg}\varphi=\operatorname{tg}\varphi_0-c_1t. \tag{43.1}$$

It is possible to establish that the solution of the variational problem for detecting the extremum of the functional, expressed in terms of the parameter of motion at the end of the powered-flight section, leads to a program determined by equation (43.1) or more general linear-fractional function

$$\operatorname{tg}\varphi=\frac{a+bt}{1+dt}.$$

In this article there is examined another problem in a more complicated setting, namely, there are considered the changeability of the field of gravitation and rotation of earth. For obtaining an optimum program it is necessary to solve the complicated system of transcendental equations, attracting numerical interation methods.

Not dwelling on this more specifically, we will say only that various examples of the numerical solution of the examined problem lead to pitch angle programs very close to the linear dependence of the pitch angle on time:

$$\varphi = \varphi_0 + \dot{\varphi} t. \qquad (43.2)$$

Depending upon basic design parameters of the rocket, the values φ_0 and $\dot{\varphi}$ assume different meanings realizing a maximum of distance.

Up till now we have talked only about results and possibilities emanating from the formulation and solution of variational problems with certain simplifying assumptions. How must we proceed with the detecting of optimum programs in real conditions?

It is necessary to consider that everything stated above no matter what without serious changes can be applied to sections of trajectory lying outside the atmosphere. If the discussion is about a single-stage rocket, then this is correct for the most final stage of the powered-flight trajectory, on which the role of the atmosphere is already insignificant. If a multistage rocket is examined, then usually this pertains to all the stages starting from the second. Possibilities of the selection of a trajectory of the first stage of a multistage rocket and the greater part of the trajectory of a single-stage rocket are rather rigidly limited by those conditions about which it was mentioned in § 41.

Thus we arrive at following rather standard scheme of selecting a program of the pitch angle:

1. Calculation is conducted of the vertical section of the trajectory up to a certain moment t_1. This time can vary with the selection of the trajectory and therefore is examined as one of the free parameters.

2. Calculation continues of the trajectory from moment t_1 under the condition that nonzero angles of attack can be allowed only up to the value of the Mach number $M = 0.7$-0.8. After that angles of attack should be close to zero during the period of the whole flight up to the moment when the influence of the atmosphere on the motion will not appear sufficiently small. Such a condition corresponds well to the dependence of the form

$$\alpha = \bar{\alpha} k (k - 2). \qquad (43.3)$$

where

$$k = 2e^{a(t_1 - t)},$$

$\bar{\alpha}$ is the limiting value of the angle of attack on the subsonic section of the trajectory, and α is a certain constant coefficient usually selected for the entire examined class of rockets. The trajectory is most sensitive to the quantity $\bar{\alpha}$, which is examined as a parameter of the family of programs.

It is easy to see that the dependence (43.3) assigns the angle of attack in the form of a curve, which rather quickly attains its maximum (in absolute magnitude) value, and then decreases, at first quickly, but with an increase in time slower and slower, tending to zero when $t \to \infty$. Coefficient a will be selected in such a manner that when $M = 0.7$-0.8 the angle of attack would already be practically equal to zero. Thus it is possible to examine the family of programs of the pitch angle dependent on two parameters: t_1 and $\bar{\alpha}$.

For single-stage rockets whose powered sections are sufficiently short, the trajectory of the maximum range is selected from such a family of two-parameter programs. The problem usually is solved on an electronic computer by means of calculation of a certain number of trajectories and detecting the extremal solution by two parameters.

If the powered section is prolonged enough so that at the end of it, after getting out of the region of intense aerodynamic action, it is possible to move again with nonzero angles of attack, then usually from some moment we pass to the program with a constant pitch angle. The basis for this is results of the solution of the variational problem for the maximum of distance under conditions of the plane-parallel field of gravity and the fact that on the powered section this field differs little from the plane-parallel.

In the examined case the program of pitch angle has the form depicted in Fig. 43.1. In the same figure there is shown the character of the change in the angle of attack. For programs obtained with every pair of values t_1 and $\bar{a}$, the magnitude of angle φ = const on the last segment is uniquely connected with time t_3 of the transition to this constant angle. Therefore time t_3 can be examined as the parameter selected with the solution of the extreme problem for the maximum of distance.

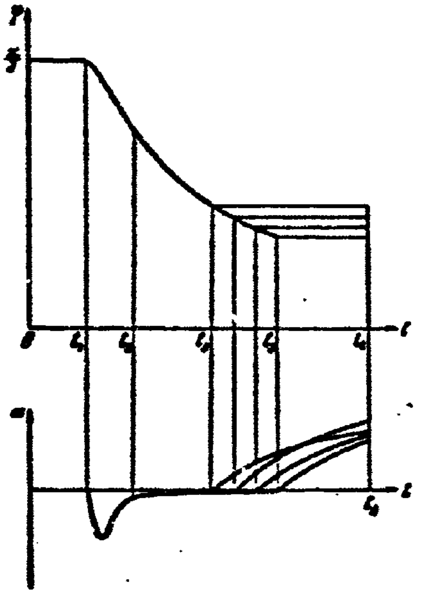

Fig. 43.1.

Thus in general the problem is reduced to a three-parameter extremal problem if there are no special conditions or limitations which determine some of these parameters independently of conditions of the maximum range. It is possible to indicate, for example, the limitations encountered in practice by maximum value of the velocity head connected either with conditions of loads on the rocket and its strength or with conditions of stabilization with limited effectiveness of the controls. There can be limitations according to the maximum permissible value of angular velocity of the turn of the rocket or the minimum permissible value of time t_1 (or the path passable on the vertical section of the flight).

With calculations of the trajectory by selection of the program of pitch angle it is best of all to use equations of motion of the form (14.25), i.e.,:

$$\frac{dv}{dt}=\frac{P-X}{m}-g\sin\theta-\frac{x}{r}g\cos\theta,$$

$$\frac{dy}{dt}=v\sin\theta,$$

$$\frac{d\theta}{dt}=\frac{1}{v}\left[\frac{\alpha}{m}\left(P+qS\frac{l_1-x_1}{l_1-x_T}c_y'\right)-g\cos\theta+\frac{x}{r}g\sin\theta\right].$$

On the vertical section the third of the equations of this system is not integrated, since

$$\theta\equiv\varphi\equiv\frac{\pi}{2}.$$

On the interval from t_1 to t_3 the angle of attack pursues in accordance with the dependence (43.3), where the above described procedure of the selection of the value a provides smallness of the angle of attack in the transonic region.

The program pitch angle is defined as sum of the assigned angle α and the angle θ obtained as a result of integration. After moment t_3, conversely, the pitch angle is assigned in the form φ = const and the dependence

$$\varphi=\theta+\alpha$$

is used as a static relation for determination of the angle of attack.

The problem of the selection of the program of pitch angle, as can be seen from the above mentioned recommendations, even for the simplest case, which is the condition of the maximum of distance for a simple single-stage rocket, in a calculating relation is quite complicated. It is necessary to conduct many monotypic calculations where the error in one of them raises doubt about certain others. Therefore, the operation manually, as a rule, occupies a very long time, requires high qualification of calculators and application of the thriftiest methods of detecting optimum values of parameters of the program. Now similar calculations are conducted only with the use of electronic computers, which permits freeing from the indicated deficiencies of manual count.

Directly for calculations of the trajectory it is necessary to follow recommendations given in § 27. Regarding methods of detecting the extremal solution, then, in general, it is possible to use any of the well-known conducting calculations. Methods of gradient or the very fastest descent are useful.

It is possible to use also the approximation of the dependence $L = f(t_1, \bar{\alpha}, t_3)$ in the form of a polynomial of the second degree:

$$L = L_0 + L'_{t_1} t_1 + L'_{\bar{\alpha}} \bar{\alpha} + L'_{t_3} t_3 + \frac{1}{2}\left(L''_{t_1 t_1} t_1^2 + L''_{\bar{\alpha}\bar{\alpha}} \bar{\alpha}^2 + L''_{t_3 t_3} t_3^2\right) + \\ + L''_{t_1 \bar{\alpha}} t_1 \bar{\alpha} + L''_{t_1 t_3} t_1 t_3 + L''_{\bar{\alpha} t_3} \bar{\alpha} t_3 \qquad (43.4)$$

ten coefficients of which are determined from the solution of the system of algebraic equations composed from results of calculations of ten trajectories with ten different combinations of parameters t_1, $\bar{\alpha}$, and t_3. Further, by equating to zero the first deviatives from distance by each of the parameters, we obtain the system of three algebraic equations:

$$\left.\begin{aligned} \frac{\partial L}{\partial t_1} &= L'_{t_1} + L''_{t_1 t_1} t_1 + L''_{t_1 \bar{\alpha}} \bar{\alpha} + L''_{t_1 t_3} t_3 = 0. \\ \frac{\partial L}{\partial \bar{\alpha}} &= L'_{\bar{\alpha}} + L''_{\bar{\alpha} t_1} t_1 + L''_{\bar{\alpha}\bar{\alpha}} \bar{\alpha} + L''_{\bar{\alpha} t_3} t_3 = 0. \\ \frac{\partial L}{\partial t_3} &= L'_{t_3} + L''_{t_3 t_1} t_1 + L''_{t_3 \bar{\alpha}} \bar{\alpha} + L''_{t_3 t_3} t_3 = 0. \end{aligned}\right\} \qquad (43.5)$$

the solution which gives the unknown values of parameters t_1, $\bar{\alpha}$, and t_3, realizing in the first approximation the maximum of range.

With detecting of the program of maximum range for a two-stage rocket we proceed approximately the same way. The difference is that in principle the quantity of parameters of the program can be increased up to any value, since limitations similar to those which were on the atmospheric section here are absent. However, on the basis of known solutions of variational problems, it is impossible to expect that programs of more complicated forms than those linearly variable with time can give substantial gain.

Not dwelling on proofs of this position, we will note only that by many calculations there is checked the impossibility of obtaining practically a noticeable gain in distance due to the complication of programs comparatively with the simple ones. However, even consideration of linear programs delivers two additional parameters, which are the initial angle $\varphi_{0\ II}$ and the speed of its change on the second step $\dot{\varphi}_{II}$, and thus the quantity of free parameters is increased to five, and together with this difficulties of a purely calculating property connected with detecting of the extremum increase. The problem in such cases is reduced to a three-parameter one, proceeding from the following considerations.

Parameter t_3 can be lowered completely, since owing to the variation of the program on the short-duration section of the trajectory between moment t_3 and the end of first stage it is difficult to reach a practical gain. The duration of the vertical section of the trajectory (up to moment t_1) is selected as small as possible, since the larger it is the steeper the trajectory (losses in speed for overcoming terrestrial gravity are increased) and the more difficult it is to achieve a turn of the speed subsequently (greater angles of attack are required).

Thus selection of the trajectory of the first stage is produced only by one parameter $\bar{\alpha}$. To each of trajectories of this family there can be applied any program in the second stage of a two-parameter family ($\varphi_{0\ II}$, $\dot{\varphi}_{II}$).

It is necessary here to make one remark. The fact is that with a similar method of composition of programs of the pitch angle, angles at the end of the first stage $\varphi_{K\ I}$ and in beginning of the second stage $\varphi_{0\ II}$ cannot be joined, and between them there can be formed breaks of greater or smaller magnitude. This will disturb point 2 of requirements for the program established in § 41. However, this disturbance will be only formal, since it is allowed only on the preliminary stage of determination of the most advantageous program. After the form and basic quantitative characteristics of the program was determined, subsequently it was "refined," i.e., acute angles on joints of neighboring sections were smoothed and "jumps" were eliminated with help of smooth transitions from one section of the program to another. With the organization of such smooth transitions we usually proceed from magnitudes of permissible angular acceleration, determined by possibilities of the system and controls.

Moving on the first part of the section of coupling with constant acceleration of one sign and on the second part of the other sign, it is possible to carry out a sufficiently smooth transition between two s-signed sections of program. as is shown on Fig. 43.2. With this the duration of the section of the program on which there is realized a jump of a given magnitude will be minimum.

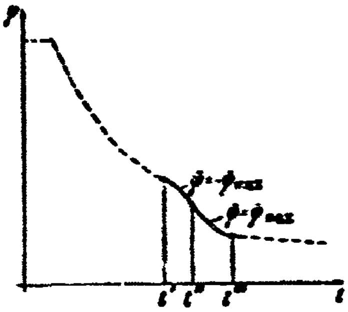

Fig. 43.2.

From everything that has been said it follows that the optimum in the sense of maximum range program for a two-stage rocket is selected from the family of three-parameter programs, where as parameters there are selected the maximum magnitude of the angle of attack $\bar{\alpha}$ on the subsonic section of the trajectory, the initial pitch angle $\varphi_{0\ II}$ and angular velocity $\dot{\varphi}_{II}$ on the second stage. This method is expedient for rockets with a number of stages greater than two. On all stages starting with the second, the pitch angle should be described by a single linear dependence of the form $\varphi = \varphi_0 - \dot{\varphi}t$. Considerable deviations from it lead only to losses of range. However, sometimes they are inevitable for the satisfaction of requirements mentioned above in point 5 § 41.

§ 44. Selection of the Program of Minimum Dispersion

Let us consider now, as is considered during the selection of the program, the condition of minimum dispersion. From expression (42.3) it is clear that with improvement of the method of turning off the engine requirements put to the program for the purpose of fulfillment of conditions of minimum dispersion are reduced, and the role of the program is as reduced.

Really, in variation of range ΔL there appear variations of kinematic parameters, induced by only the deviation $\delta\varphi(t)$ of the program of pitch from the nominal. But the perfected control system strives to turn the variation ΔL into zero independently what is the cause of the appearance of this variation.

It is possible to imagine the method of turning off of the engine founded on the measurement of all six parameters of motion and continuous calculation with the help of a special flying range computer:

$$L = f(v_x, v_y, v_z, x, y, z) \qquad (44.1)$$

or

$$L = f(x_i), \qquad i = 1, 2, \ldots, 6.$$

where $x_1, \ldots, x_6$ are any six quantities connected one-to-one with $v_x, \ldots, z$, which can be measured by the range control system. When a given function attains an assigned value, the command for turning off the engine is sent.

Obviously, in this case methodical errors, including those induced by deviation $\delta\varphi(t)$, will be reduced to zero, and deviations in range will appear only as a result of instrumental measuring errors of parameters of motion. Correct to linear members the range error will be equal to

$$\Delta L = \frac{\partial L}{\partial v_x}\Delta v_{x\,и} + \frac{\partial L}{\partial v_y}\Delta v_{y\,и} + \frac{\partial L}{\partial v_z}\Delta v_{z\,и} + \frac{\partial L}{\partial x}\Delta x_и + \\ + \frac{\partial L}{\partial y}\Delta y_и + \frac{\partial L}{\partial z}\Delta z_и \qquad (44.2)$$

or

$$\Delta L = \sum_{i=1}^{6} \frac{\partial L}{\partial x_i}\Delta x_{i\,и}.$$

where $\Delta v_{xи}$, $\Delta v_{yи}$, ..., $\Delta z_и$ or $\Delta x_{1и}$ are instrumental errors of measurements of corresponding parameters.

The influence of the program of the pitch angle on range dispersion in the examined case will appear in terms of derivatives $\partial L/\partial v_x$, $\partial L/\partial v_y$, ..., $\partial L/\partial z$, dependent on the computed values of parameters of motion at the time of turning off the engine. Therefore, in principle with the help of the selection of the program it is possible to minimize the magnitude of the standard deviation in range. This deviation, if one were to consider instrumental errors $\Delta v_{xи}$, ..., $\Delta z_и$ ($\Delta x_{1и}$) random and independent and designate the corresponding mean quadratic errors of measurements σv_x, ..., σz (σx_1), it is possible to record in the form

$$\sigma L = \sqrt{\left(\frac{\partial L}{\partial v_x}\sigma v_{x\,и}\right)^2 + \left(\frac{\partial L}{\partial v_y}\sigma v_{y\,и}\right)^2 + \cdots + \left(\frac{\partial L}{\partial z}\sigma z_и\right)^2} \qquad (44.3)$$

or

$$\sigma L = \sqrt{\sum_{i=1}^{6}\left(\frac{\partial L}{\partial x_i}\sigma x_{i\,и}\right)^2}.$$

Derivatives of distance with respect to parameters of motion should be considered functions of parameters of the program, and such values of the latter, which reduce the value of expression (44.3) to a minimum, should be found.

The formula for turning off the engine (44.1) can be presented in another form, decomposing the function in Taylor series in the vicinity of the calculation point in powers of deviations of parameters of motion from computed values:

$$\Delta L = \sum_{i=1}^{6}\left(\frac{\partial L}{\partial x_i}\Delta x_i + \frac{1}{2}\frac{\partial^2 L}{\partial x_i^2}\Delta x_i^2\right) + \\ + \sum_{\substack{i,j=1\\ i<j}}^{6} \frac{\partial^2 L}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \ldots = 0. \qquad (44.4)$$

If in formula (44.4) we are limited by the final number of terms of expansion, for example, only by linear members, then besides instrumental there error will appear methodical errors in the form of the sum of rejected members. It is clear that with the help of the program of pitch angle it is possible to influence not only the instrumental errors but also the methodical, using the dependence of coefficients of formula (44.4) (i.e., partial derivatives) from the program. But here another circumstance is important, which occurs in the dependence of methodical errors on quantities Δx_i which in the end are determined by having an effect on the rocket in flight by random perturbations. Methods of the determination of the influence on the trajectory of small deviations of design parameters and certain other causes were discussed in Chapter IX. Using these methods, one can determine for an assigned totality of random independent perturbations and for the assigned program of pitch angle the deviations in range under the condition of the turning off of the engine by the assigned formula. This will give to us methodical errors in range.

Thus the problem is reduced to the selection of parameters of the program from the condition of the minimum of the total standard deviation in range owing to both methodical and instrumental errors of control.

The described approach to the selection of the program of minimum dispersion is quite common and is useful for any methods of turning off of the engine. Let us dwell more concretely on methods connected with the application of integrators of G-forces in different variants.

For the case when the turning off of the engine is produced from the integrator of axial G-forces, the formulas for the determination of methodical and instrumental errors were obtained in § 36 and 37. The equation of the operation of the simplest integrator

$$v_x = v\cos\alpha + \int_0^t g\sin\varphi\,dt + \int_0^t v\dot{\varphi}\sin\alpha\,dt$$

does not contain any coefficients, selecting which properly it would have been possible to affect methodical errors in range. Thus both methodical and instrumental errors are only functions of parameters of the pitch angle program. It is assumed, of course, that the probability characteristics (in the first place standard deviation) of the random error of measurement of apparent speed are the given value. Thus scatterings of perturbing factors are assigned. We see that the problem is reduced to the determination of values of a certain quantity of parameters of the program from the condition of the minimum of total deviation in range.

The integrator of axial G-forces with temporary compensation permits ordering one more value, namely, the coefficient of compensation. For every program determined by the totality of some quantity of its parameters, it is possible to examine the turning off of the engine at different values of the coefficient of compensation, but with assigned probability characteristics of instrumental errors and perturbing causes. The value of the coefficient of compensation with which will be realized the minimum error in range will be optimum for a given program.

Inasmuch as such a relative minimum exists for every program of the examined family, it is necessary to select that program and that coefficient which give absolute minimum of deviation in range. However, the selection performed by the described method will give the best result only for some one ... moment of turning off of the engine, i.e., for some one range. Thus, theoretically it would be necessary to have an infinite quantity of programs and selected coefficients of compensation respectively. In practice we manage with a small quantity of programs covering a whole range of ranges. Of course, it is necessary for some ranges to retreat from conditions of providing the minimum possible dispersion.

Approximately the same way is the matter with the selection of programs in the case of turning off of the engine from the integrator with a constant inclination of the axis of sensitivity, i.e., installed on the stabilized platform. Here there is examined the problem on the minimization of total deviation in range owing to the definite quantity of parameters properly of the program and angle of inclination of the axis of sensitivity of the integrator.

If the controlling functional is complicated by the introduction of double integration of the G-force, then with minimization of deviations in range the direction of the axis of sensitivity along which calculation of the apparent path is produced will also be subject to the determination.

It is not difficult to see that almost in all problems minimization should be produced according to the quantity of parameters fluctuating from one to five. Since the strict solution is sometimes hampered even with the application of electronic computers, it is possible preliminarily to conduct an analysis of the dispersion. With this instrumental errors depending upon the program and methodical errors are examined separately for more or less suitable programs depending upon coefficients of the controlling functional (coefficient of compensation, directions of axes of sensitivity, and so forth). In § 37 it was shown that optimum values of the indicated coefficients from the condition of the minimum of methodical error are determined quite simply. This occurs, as a rule, sufficiently in order to dwell on some narrow beam of programs and formulate a concept about basic regularities which are obeyed certain components of the total deviation in range.

By conducting similar calculations for the upper and lower boundaries of the assigned range of distances and also for one-two intermediate points, there can be made a selection of both the number of programs and the numerical values of parameters of these programs. It is necessary to remember that programs of both minimum dispersion and maximum range essentially depend on the direction of firing and latitude of the launching point. With firing to the east the effect of rotation of the earth (Coriolis acceleration and turn of gyroscopes expressed by the angle γ_3) as if lifts the trajectory, makes it steeper, and with firing to the west, conversely, the trajectory is as if pressed to the earth and becomes more sloping. Therefore programs of the pitch angle in the first case should place the rocket at smaller angles of inclination of the tangent to the trajectory and in the second case, at larger angles. It is natural that this effect is increased with a decrease in latitude of the point of launching.

Everything that has been said does not exhaust all the problems connected with selection of the program but gives an approach to the solution of problem and fixes attention to the most essential sides of the problem.

GRAPHS FOR DESIGN CALCULATIONS

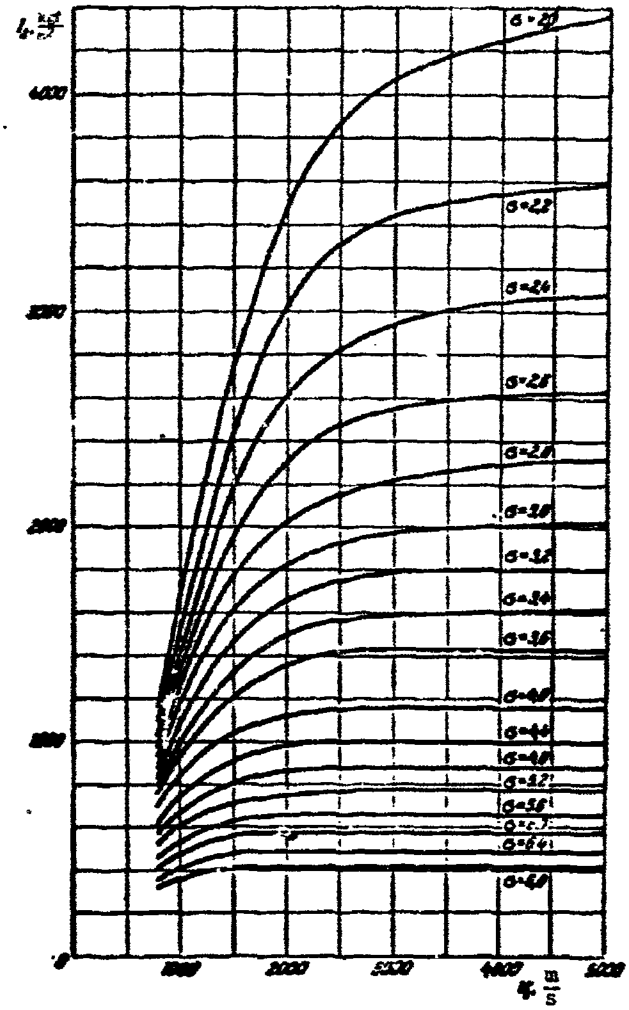

Fig. I. Graph of the dependence of integral I_2 from v_1 and σ.

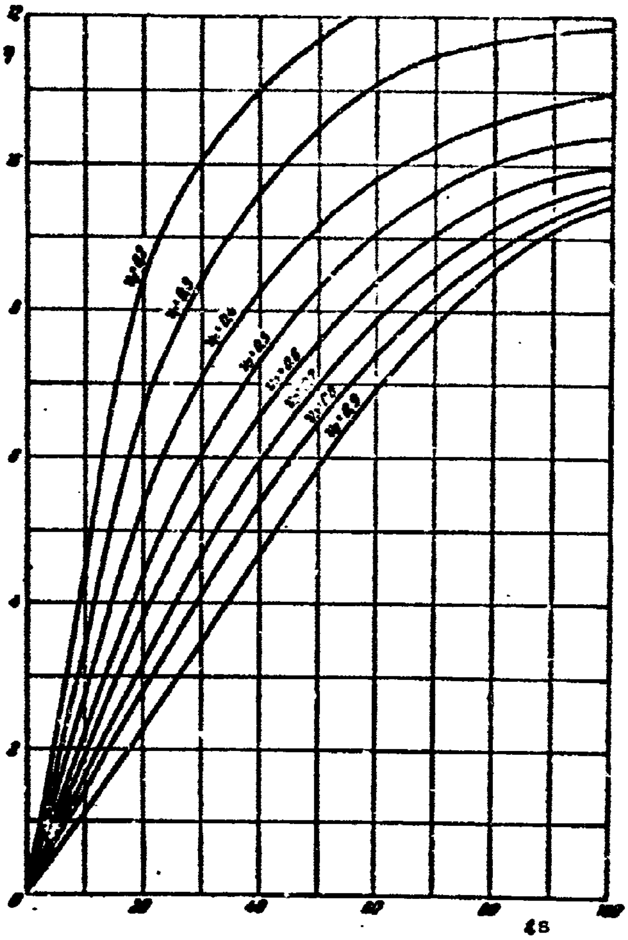

Fig. 'I. Graph of the change in η depending upon t and v_0.

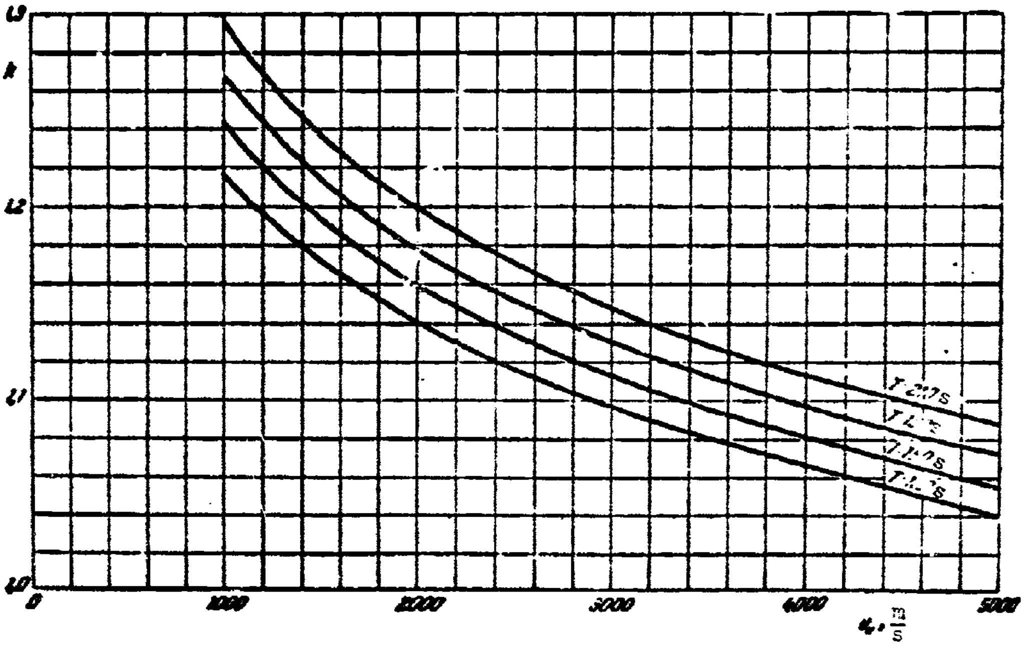

Fig. III. Graph of the change in coefficient k depending upon $v_к$ and T.

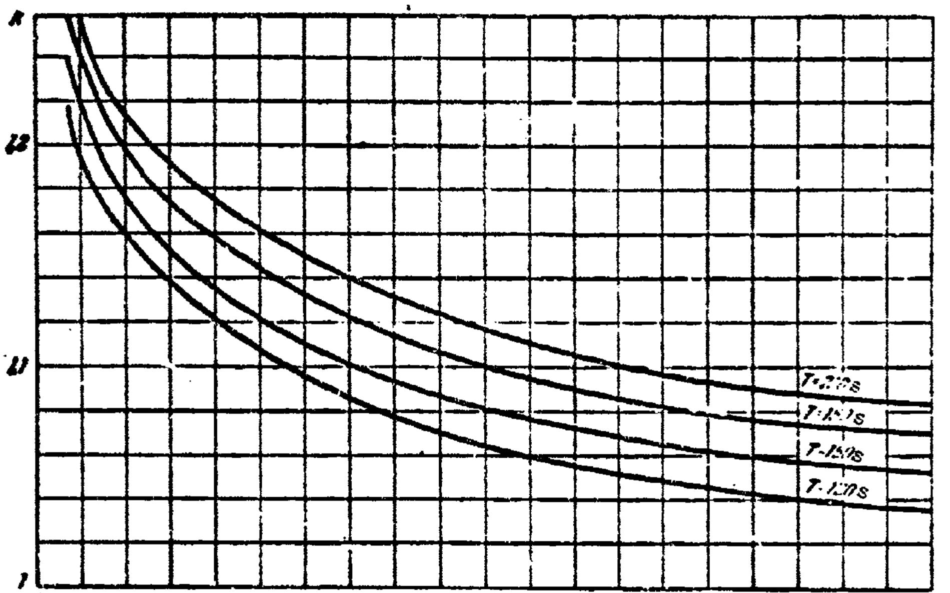

Fig. IV. Graph of the change in coefficient k depending on L and T.

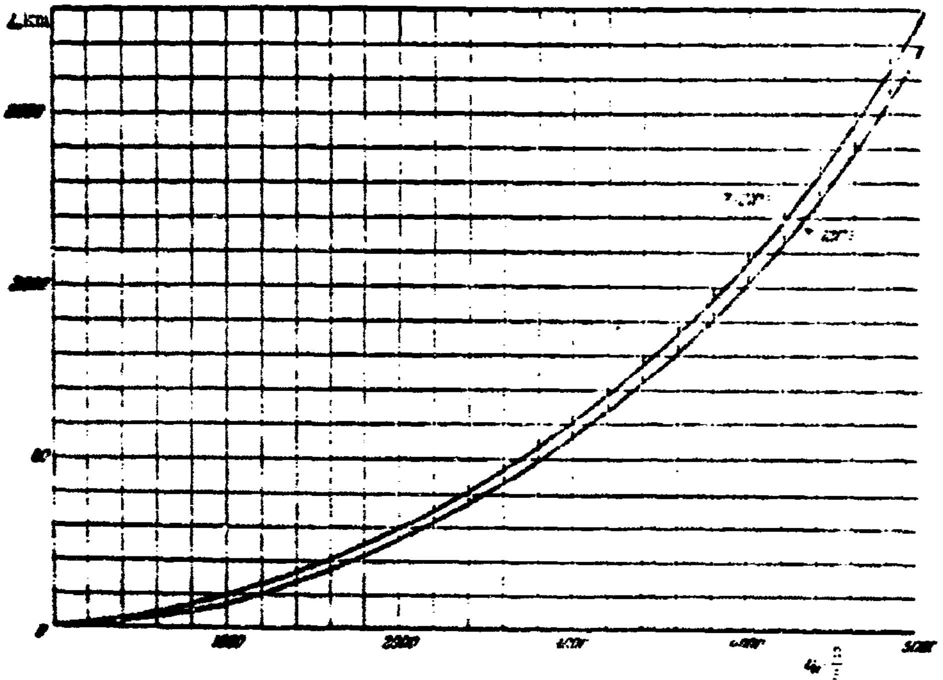

Fig. V. Graph of the change in L depending upon [illegible]

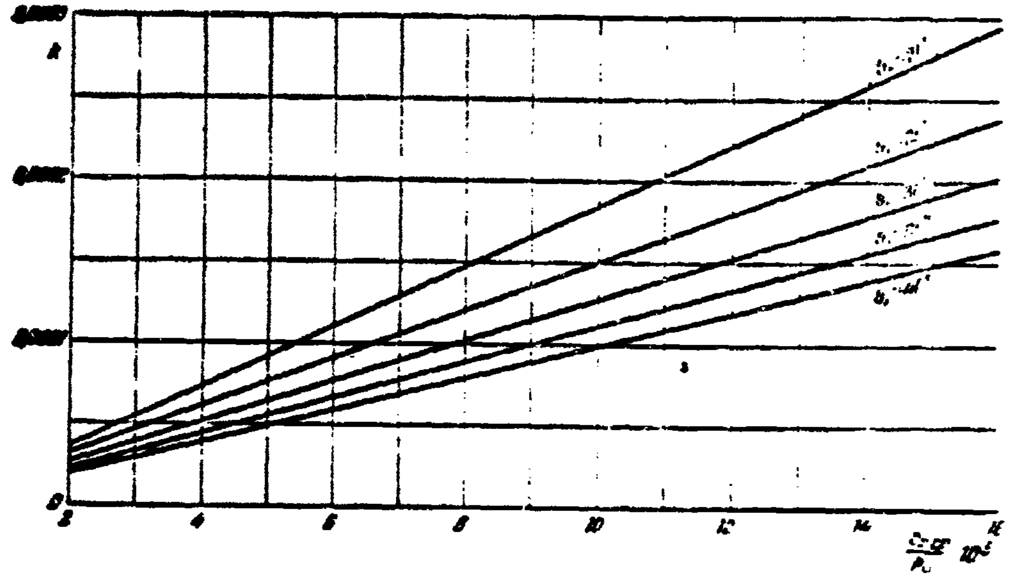

Fig. VI. Graph of the change in coefficient k depending upon θ_1 and relation C_{xcp}/P_1.

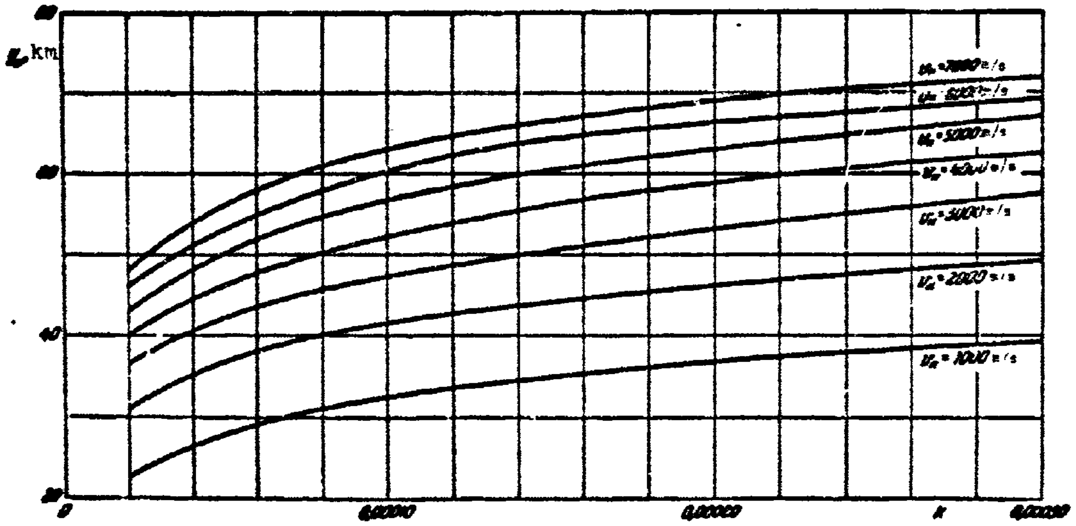

Fig. VII. Graph of the change in y_H depending upon k and v_K.

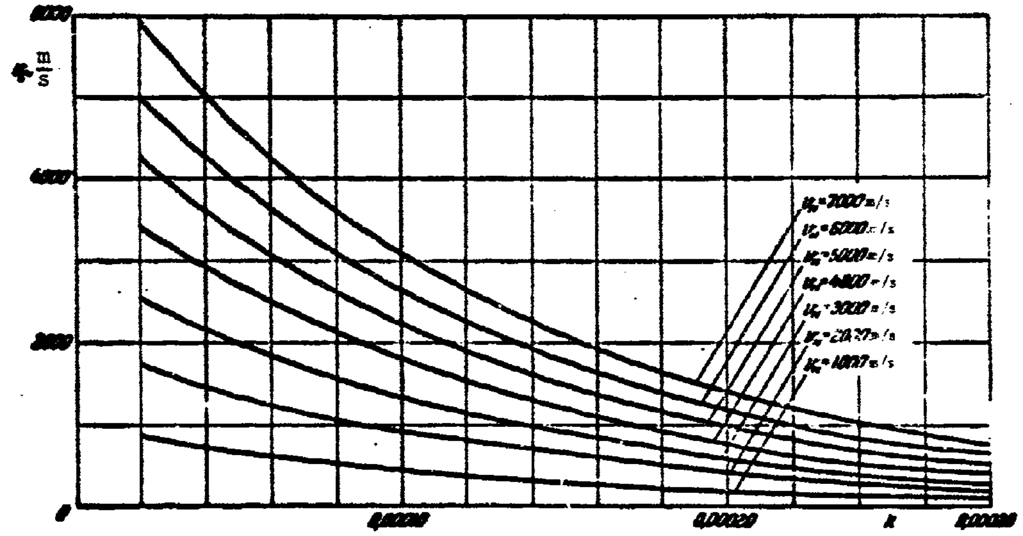

Fig. VIII: Graph of the change in v_C depending upon k and v_H.

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О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

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rot	curl
lg	log